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Study Manual for SOA Exam FAM-L

1st Edition, 2nd Printing

by

Johnny Li, Ph.D., FSA

Andrew Ng, Ph.D., FSA

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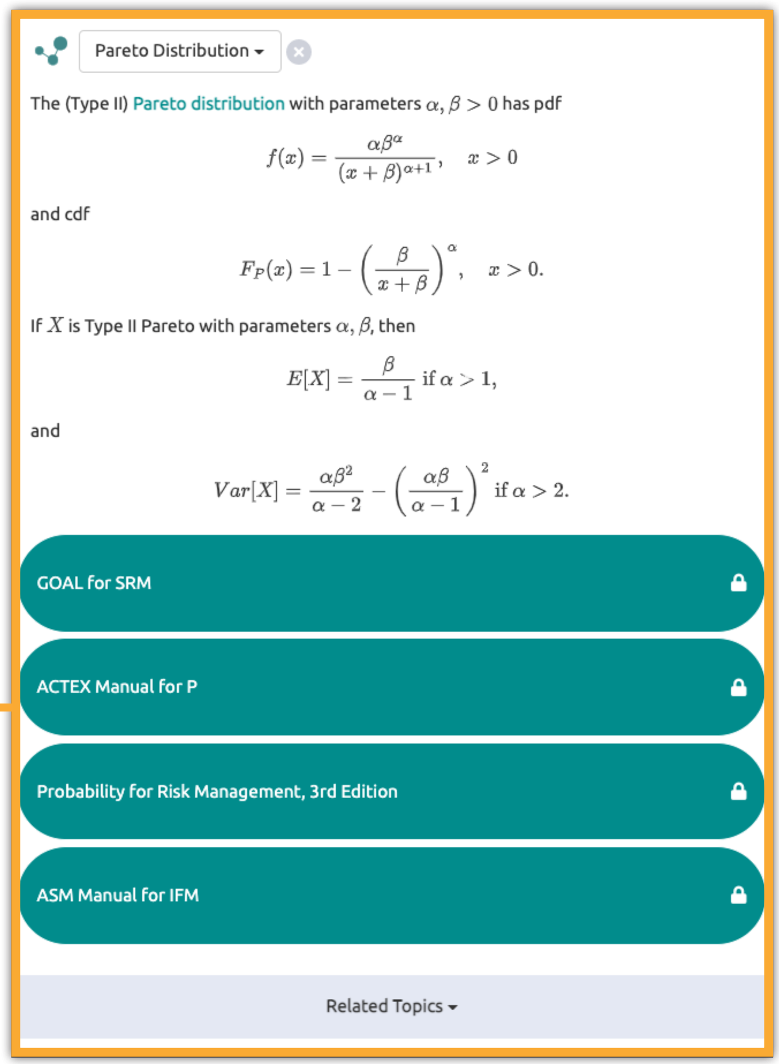

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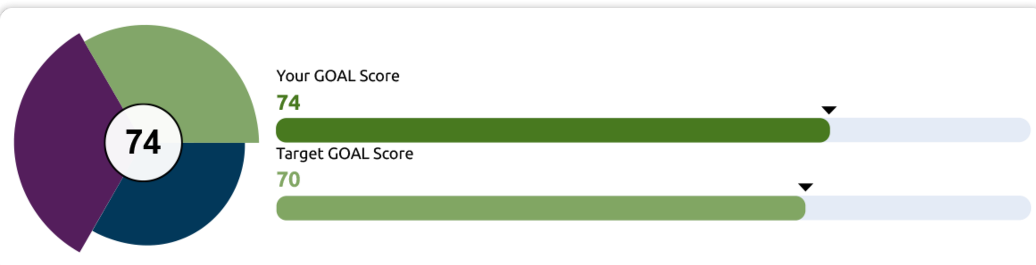

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05/24/2022 21:57:43	05/24/2022 21:57:43		Quiz	Continuous P...	25	New	Resume
05/24/2022 10:34:05	05/24/2022 14:57:49	05/24/2022 14:57:49	Practice Session	Addition and ...	80	Complete	Review
05/21/2022 19:32:50	05/23/2022 20:02:00		Simulated Exam	Exam 2	30	Reviewing	Complete
05/17/2022 15:19:19	05/17/2022 15:46:03	05/23/2022 14:15:29	Simulated Exam	Exam 6	30	Complete	Review
05/14/2022 11:26:59	05/14/2020 12:02:36	05/23/2022 11:57:47	Quiz	Conditional D...	20	Complete	Review

Quickly return to previous sessions.

Contents

Chapter 2

Life Tables

Objectives

1. To apply life tables
2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality
3. To calculate moments for future lifetime random variables
4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2.1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions l_x and d_x, and explain how they are applied.

x	l_x	d_x
0	1000	16
1	984	7
2	977	12
3	965	75
4	890	144

In this hypothetical life table, the value of l_0 is 1,000. This starting value is called the radix of the life table. For $x = 1, 2, \ldots,$ the function l_x stands for the expected number of persons who can survive to age x. Given an assumed value of l_0, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0 S_0(x).$$

In the other way around, given the life table function l_x, we can easily obtain values of $S_0(x)$ for integral values of x using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$

Furthermore, we have

$$ {}_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}/l_0}{l_x/l_0} = \frac{l_{x+t}}{l_x}, $$

which means that we can calculate ${}_tp_x$ for all integral values of t and x from the life table function l_x.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by ${}_td_x$. It immediately follows that ${}_td_x = l_x - l_{x+t}$.

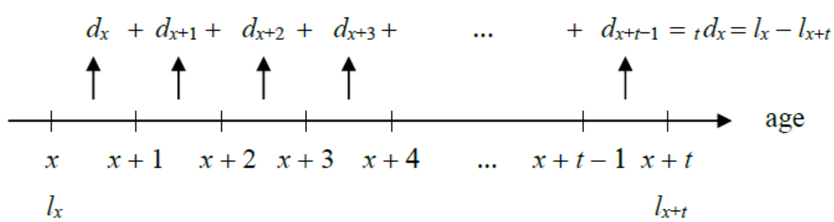
We can then calculate ${}_tq_x$ and ${}_{m|n}q_x$ by the following two relations:

$$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \quad {}_{m|n}q_x = \frac{{}_nd_{x+m}}{l_x} = \frac{l_{x+m} - l_{x+m+n}}{l_x}. $$

When $t = 1$, we can omit the subscript t and write ${}_1d_x$ as d_x. By definition, we have

$$ {}_td_x = d_x + d_{x+1} + \cdots + d_{x+t-1}. $$

Graphically,



Also, when $t = 1$, we have the following relations:

$$ d_x = l_x - l_{x+1}, \ p_x = \frac{l_{x+1}}{l_x}, \text{ and } q_x = \frac{d_x}{l_x}. $$

Summing up, with the life table functions l_x and d_x, we can recover survival probabilities ${}_tp_x$ and death probabilities ${}_tq_x$ for all integral values of t and x easily.

Life Table Functions

(2.1) $$ {}_tp_x = \frac{l_{x+t}}{l_x} $$

(2.2) $$ {}_td_x = l_x - l_{x+t} = d_x + d_{x+1} + \cdots + d_{x+t-1} $$

(2.3) $$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x} $$

Some FAM-L exam and LTAM (the predecessor of FAM-L) questions are based on the Standard Ultimate Life Table, and some MLC (the predecessor of LTAM) exam questions are based on the Illustrative Life Table. The Standard Ultimate Life Table can be found at the SOA's website:

https://www.soa.org/Education/Exam-Req/edu-exam-ltam-detail.aspx,

and the Illustrative Life Table is provided in Appendix 2 of this study manual. The two tables have very similar formats. They contain a lot of information. For now, you only need to know and use the first three columns: x, l_x, and q_x (Standard Ultimate Life Table) and $1000q_x$ (Illustrative Life Table). For example, to obtain q_{43}, simply use the column labeled q_x. You should obtain $q_{43} = 0.000656$ (from the Standard Ultimate Life Table). It is also possible, but more tedious, to calculate q_{43} using the column labeled l_x; we have $q_{43} = 1 - l_{44}/l_{43} = 1 - 99104.3/99169.4 = 0.000656452$.

To get values of ${}_tp_x$ and ${}_tq_x$ for $t > 1$, you should always use the column labeled l_x. For example, we have ${}_5p_{61} = l_{66}/l_{61} = 94020.3/96305.8 = 0.976268$ and ${}_5q_{61} = 1 - {}_5p_{61} = 1 - 0.976268 = 0.023732$ (from the Standard Ultimate Life Table). Here, you should not base your calculations on the column labeled q_x, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

Example 2.1.

You are given the following excerpt of a life table:

x	l_x	d_x
20	96178.01	99.0569
21	96078.95	102.0149
22	95976.93	105.2582
23	95871.68	108.8135
24	95762.86	112.7102
25	95650.15	116.9802

Calculate the following:

(a) ${}_5p_{20}$

(b) q_{24}

(c) ${}_{4|1}q_{20}$

Solution:

(a) $${}_5p_{20} = \frac{l_{25}}{l_{20}} = \frac{95650.15}{96178.01} = 0.994512.$$

(b) $$q_{24} = \frac{d_{24}}{l_{24}} = \frac{112.7102}{95762.86} = 0.001177.$$

(c) $${}_{4|1}q_{20} = \frac{{}_1d_{24}}{l_{20}} = \frac{112.7102}{96178.01} = 0.001172.$$

□

Example 2.2.

You are given:

(i) $S_0(x) = 1 - \dfrac{x}{100}, \quad 0 \le x \le 100$

(ii) $l_0 = 100$

(a) Find an expression for l_x for $0 \le x \le 100$.

(b) Calculate q_2.

(c) Calculate ${}_3q_2$.

Solution:

(a) $l_x = l_0 S_0(x) = 100 - x$.

(b) $q_2 = \dfrac{l_2 - l_3}{l_2} = \dfrac{98 - 97}{98} = \dfrac{1}{98}$.

(c) ${}_3q_2 = \dfrac{l_2 - l_5}{l_2} = \dfrac{98 - 95}{98} = \dfrac{3}{98}$.

□

In Exam FAM-L, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3.

For a certain population of 20 year olds, you are given:

(i) 2/3 of the population are nonsmokers, and 1/3 of the population are smokers.

(ii) The future lifetime of a nonsmoker is uniformly distributed over [0, 80).

(iii) The future lifetime of a smoker is uniformly distributed over [0, 50).

Calculate ${}_5p_{40}$ for a life randomly selected from those surviving to age 40.

Solution: The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 20.

- Suppose that there are l_{20} persons in the entire population initially. At time 0 (i.e., at age 20), there are $\frac{2}{3}\,l_{20}$ nonsmokers and $\frac{1}{3}\,l_{20}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 40 is $1 - 20/80 = 3/4$. For smokers, the proportion of individuals who can survive to age 40 is $1 - 20/50 = 3/5$. At age 40, there are $\frac{3}{4}\frac{2}{3}\,l_{20} = 0.5 l_{20}$ nonsmokers and $\frac{3}{5}\frac{1}{3}\,l_{20} = 0.2 l_{20}$ smokers. Hence, among those who can survive to age 40, 5/7 are nonsmokers and 2/7 are smokers.

Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over $[0, 60)$. This means that the probability for a nonsmoker to survive from age 40 to age 45 is $1 - 5/60 = 11/12$.
- For a smoker at age 40, the remaining lifetime is uniformly distributed over $[0, 30)$. This means that the probability for a smoker to survive from age 40 to age 45 is $1 - 5/30 = 5/6$.

Finally, for the whole population, we have

$$ {}_5p_{40} = \frac{5}{7} \times \frac{11}{12} + \frac{2}{7} \times \frac{5}{6} = \frac{25}{28}. $$

□

2.2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of ${}_tp_x$ and ${}_tq_x$ when both t and x are integers. But what if t and/or x are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a fractional age assumption.

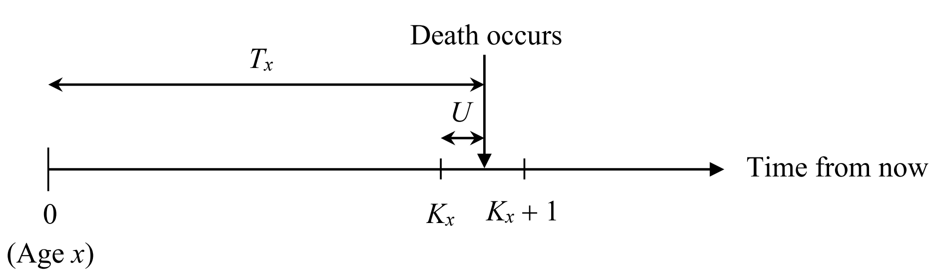
In Exam FAM-L, you are required to know two fractional age assumptions:

1. Uniform distribution of death
2. Constant force of mortality

We go through these assumptions one by one.

Assumption 1: Uniform Distribution of Death

The Uniform Distribution of Death (UDD) assumption is extensively used in the Exam FAM-L syllabus. The idea behind this assumption is that we use a bridge, denoted by U, to connect the (continuous) future lifetime random variable T_x and the (discrete) curtate future lifetime random variable K_x. The idea is illustrated diagrammatically as follows:



It is assumed that U follows a uniform distribution over the interval [0, 1], and that U and K_x are independent. Then, for $0 \le r < 1$ and an integral value of x, we have

$$\begin{aligned} {}_rq_x &= \Pr(T_x \le r) \\ &= \Pr(U < r \cap K_x = 0) \\ &= \Pr(U < r)\Pr(K_x = 0) \\ &= rq_x. \end{aligned}$$

The second last step follows from the assumption that U and K_x are independent, while the last step follows from the fact that U follows a uniform distribution over [0, 1].

Key Equation for the UDD Assumption

(2.4) $${}_rq_x = rq_x, \quad \text{for } 0 \le r < 1$$

This means that under UDD, we have, for example, ${}_{0.4}q_{50} = 0.4q_{50}$. The value of q_{50} can be obtained straightforwardly from the life table. To calculate ${}_rp_x$, for $0 \le r < 1$, we use ${}_rp_x = 1 - {}_rq_x = 1 - rq_x$. For example, we have ${}_{0.1}p_{20} = 1 - 0.1q_{20}$.

Equation (2.4) is equivalent to a linear interpolation between l_x and l_{x+1}, that is,

$$l_{x+r} = (1-r)l_x + rl_{x+1}.$$

Proof:

$$\begin{aligned} {}_rp_x &= 1 - rq_x = (1-r) + rp_x \\ \frac{l_{x+r}}{l_x} &= (1-r) + r\frac{l_{x+1}}{l_x} \\ l_{x+r} &= (1-r)l_x + rl_{x+1} \end{aligned}$$

□

You will find this equation – the interpolation between l_x and l_{x+1} – very useful if you are given a table of l_x (instead of q_x).

Application of the UDD Assumption to l_x

(2.5) $$l_{x+r} = (1-r)l_x + rl_{x+1}, \quad \text{for } 0 \le r < 1$$

What if the subscript on the left-hand-side of ${}_rq_x$ is greater than 1? In this case, we should first use equation (1.6) from Chapter 1 to break down the probability into smaller portions. As an example, we can calculate ${}_{2.5}p_{30}$ as follows:

$${}_{2.5}p_{30} = {}_2p_{30} \times {}_{0.5}p_{32} = {}_2p_{30} \times (1 - 0.5q_{32}).$$

The value of ${}_2p_{30}$ and q_{32} can be obtained from the life table straightforwardly.

What if the subscript on the right-hand-side is not an integer? In this case, we should make use of a special trick, which we now demonstrate. Let us consider ${}_{0.1}p_{5.7}$ (both subscripts are not integers). The trick is that we multiply this probability with ${}_{0.7}p_5$, that is,

$${}_{0.7}p_5 \times {}_{0.1}p_{5.7} = {}_{0.8}p_5.$$

This gives ${}_{0.1}p_{5.7} = \dfrac{{}_{0.8}p_5}{{}_{0.7}p_5} = \dfrac{1-0.8q_5}{1-0.7q_5}$. The value of q_5 can be obtained from the life table.

To further illustrate this trick, let us consider ${}_{3.5}p_{4.6}$: This probability can be evaluated from the following equation:

$$ {}_{0.6}p_4 \times {}_{3.5}p_{4.6} = {}_{4.1}p_4. $$

Then, we have ${}_{3.5}p_{4.6} = \dfrac{{}_{4.1}p_4}{{}_{0.6}p_4} = \dfrac{{}_4p_4\ {}_{0.1}p_8}{{}_{0.6}p_4} = \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$, and finally ${}_{3.5}q_{4.6} = 1 - \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$.

The values of ${}_4p_4$, q_8 and q_4 can be obtained from the life table.

Let us study the following example.

Example 2.4. You are given the following excerpt of a life table:

x	l_x	d_x
60	100000	300
61	99700	400
62	99300	500
63	98800	600
64	98200	700
65	97500	800

Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) ${}_{0.26}p_{61}$

(b) ${}_{2.2}q_{60}$

(c) ${}_{0.3}q_{62.8}$

Solution:

(a) ${}_{0.26}p_{61} = 1 - {}_{0.26}q_{61} = 1 - 0.26 \times 400/99700 = 0.998957.$

Alternatively, we can calculate the answer by using a linear interpolation between l_{61} and l_{62} as follows:

$$ l_{61.26} = (1-0.26)l_{61} + 0.26l_{62} = 0.74 \times 99700 + 0.26 \times 99300 = 99596. $$

It follows that ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99596/99700 = 0.998957.$

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_2p_{60} \times {}_{0.2}p_{62} = 1 - {}_2p_{60} \times (1 - 0.2q_{62})$

$$ = 1 - \frac{l_{62}}{l_{60}}\left(1 - 0.2 \times \frac{d_{62}}{l_{62}}\right) = 1 - \frac{99300}{100000}\left(1 - 0.2 \times \frac{500}{99300}\right) = 0.008. $$

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} as follows:

$$ l_{62.2} = (1-0.2)l_{62} + 0.2l_{63} = 0.8 \times 99300 + 0.2 \times 98800 = 99200. $$

It follows that ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 1 - 99200/100000 = 0.008.$

(c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute ${}_{0.3}p_{62.8}$ from the following equation:

$$ {}_{0.8}p_{62} \times {}_{0.3}p_{62.8} = {}_{1.1}p_{62}. $$

Trick

Rearranging the equation above, we have

$$ {}_{0.3}p_{62.8} = \frac{{}_{1.1}p_{62}}{{}_{0.8}p_{62}} = \frac{p_{62}\ {}_{0.1}p_{63}}{{}_{0.8}p_{62}} = \frac{p_{62}\,(1-0.1q_{63})}{1-0.8q_{62}} = \frac{\dfrac{98800}{99300}\left(1-0.1\times\dfrac{600}{98800}\right)}{1-0.8\times\dfrac{500}{99300}} $$

$$ = 0.998382. $$

Hence, ${}_{0.3}q_{62.8} = 1 - 0.998382 = 0.001618$.

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} and another interpolation between l_{63} and l_{64}:

First,

$$ l_{62.8} = (1-0.8)l_{62} + 0.8l_{63} = 0.2 \times 99300 + 0.8 \times 98800 = 98900. $$

Second,

$$ l_{63.1} = (1-0.1)l_{63} + 0.1l_{64} = 0.9 \times 98800 + 0.1 \times 98200 = 98740. $$

Finally,

$$ {}_{0.3}q_{62.8} = 1 - {}_{0.3}p_{62.8} = 1 - l_{63.1}/l_{62.8} = 1 - 98740/98900 = 0.001618. $$

□

Sometimes, you may be asked to calculate the density function of T_x and the force of mortality from a life table. Under UDD, we have the following equation for calculating the density function:

$$ f_x(r) = q_x, \quad 0 \le r < 1. $$

Proof: $f_x(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}F_x(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}\Pr(T_x \le r) = \dfrac{\mathrm{d}}{\mathrm{d}r}{}_rq_x = \dfrac{\mathrm{d}}{\mathrm{d}r}(rq_x) = q_x.$ □

Under UDD, we have the following equation for calculating the force of mortality:

$$ \mu_{x+r} = \frac{q_x}{1-rq_x}, \quad 0 \le r < 1. $$

Proof: In general, $f_x(r) = {}_rp_x\mu_{x+r}$. Under UDD, we have $f_x(r) = q_x$ and ${}_rp_x = 1 - rq_x$. The result follows. □

Let us take a look at the following example.

Example 2.5. **[Course 3 Spring 2000 #12]**

For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$

(ii) $\mu_{81.5} = 0.0408$

(iii) $\mu_{82.5} = 0.0619$

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

Solution: The probability that a person age 80.5 will die within two years is ${}_2q_{80.5}$. We have

$$_{0.5}p_{80} \times {}_2p_{80.5} = {}_{2.5}p_{80}.$$

This gives

$$_2p_{80.5} = \frac{_2p_{80}\ {}_{0.5}p_{82}}{_{0.5}p_{80}} = \frac{p_{80}p_{81}\left(1-0.5q_{82}\right)}{1-0.5q_{80}} = \frac{\left(1-q_{80}\right)\left(1-q_{81}\right)\left(1-0.5q_{82}\right)}{1-0.5q_{80}}.$$

We then need to find q_{80}, q_{81} and q_{82} from the information given in the question. Using $\mu_{80.5}$, we have $\mu_{80.5} = \dfrac{q_{80}}{1-0.5q_{80}} \Rightarrow q_{80} = 0.0200$. Similarly, by using $\mu_{81.5}$ and $\mu_{82.5}$, we obtain $q_{81} = 0.0400$ and $q_{82} = 0.0600$.

Substituting q_{80}, q_{81} and q_{82}, we obtain ${}_2p_{80.5} = 0.921794$, and hence ${}_2q_{80.5} = 1 - {}_2p_{80.5} = 0.0782$. Hence, the answer is (A).

□

Assumption 2: Constant Force of Mortality

The idea behind this assumption is that for every age x, we approximate μ_{x+k} for $0 \leq r < 1$ by a constant, which we denote by $\tilde{\mu}_x$. This means

$$\int_0^1 \mu_{x+u}\ \mathrm{d}u = \int_0^1 \tilde{\mu}_x\ \mathrm{d}u = \tilde{\mu}_x,$$

which implies $p_x = e^{-\tilde{\mu}_x}$ and $\tilde{\mu}_x = -\ln\left(p_x\right)$.

We are now ready to develop equations for calculating various death and survival probabilities. First of all, for any integer-valued x, we have

$$_rp_x = \left(p_x\right)^r, \quad 0 \leq r < 1.$$

Proof: $_rp_x = \exp\left(-\int_0^r \mu_{x+u}\mathrm{d}u\right) = \exp\left(-\int_0^r \tilde{\mu}_x\mathrm{d}u\right) = e^{-\tilde{\mu}_x r} = \left(e^{-\tilde{\mu}_x}\right)^r = \left(p_x\right)^r.$ □

For example, ${}_{0.3}p_{50} = \left(p_{50}\right)^{0.3}$, and ${}_{0.4}q_{62} = 1 - {}_{0.4}p_{62} = 1 - \left(p_{62}\right)^{0.4}$. We can generalize the equation above to obtain the following key formula.

Key Equation for the Constant Force of Mortality Assumption

(2.6) $${}_rp_{x+u} = (p_x)^r, \quad \text{for } 0 \le r < 1 \text{ and } r+u \le 1$$

Proof: ${}_rp_{x+u} = \exp\left(-\int_0^r \mu_{x+u+t}\mathrm{d}t\right) = \exp\left(-\int_0^r \tilde{\mu}_x\mathrm{d}t\right) = e^{-\tilde{\mu}_x r} = (p_x)^r$.

[The second step follows from the fact that given $0 \le r < 1$, $u+t$ is always less than or equal to 1 when $0 \le t \le r$.] □

Notice that the key equation for the constant force of mortality assumption is based on p, while that for the UDD assumption is based on q.

This key equation means that, for example, ${}_{0.2}p_{30.3} = (p_{30})^{0.2}$. Note that the subscript u on the right-hand-side does not appear in the result, provided that the condition $r+u \le 1$ is satisfied. But what if $r+u > 1$? The answer is very simple: Split the probability! To illustrate, let us consider ${}_{0.8}p_{30.3}$. (Here, $r+u = 0.8+0.3 = 1.1 > 1$.) By using equation (1.6) from Chapter 1, we can split ${}_{0.8}p_{30.3}$ into two parts as follows:

$${}_{0.8}p_{30.3} = {}_{0.7}p_{30.3} \times {}_{0.1}p_{31}.$$

We intentionally consider a duration of 0.7 years for the first part, because $0.3+0.7=1$, which means we can apply the key equation ${}_rp_{x+u} = (p_x)^r$ to it. As a result, we have

$${}_{0.8}p_{30.3} = (p_{30})^{0.7} \times (p_{31})^{0.1}.$$

The values of p_{30} and p_{31} can be obtained from the life table straightforwardly.

To further illustrate, let us consider ${}_{5.6}p_{40.8}$. We can split it as follows:

$${}_{5.6}p_{40.8} = {}_{0.2}p_{40.8} \times {}_{5.4}p_{41} = {}_{0.2}p_{40.8} \times {}_5p_{41} \times {}_{0.4}p_{46} = (p_{40})^{0.2} \times {}_5p_{41} \times (p_{46})^{0.4}.$$

The values of p_{40}, ${}_5p_{41}$ and p_{46} can be obtained from the life table straightforwardly.

Interestingly, equation (2.6) implies that for $0 \le r < 1$, the value of $\ln(l_{x+r})$ can be obtained by a linear interpolation between the values of $\ln(l_x)$ and $\ln(l_{x+1})$.

Proof: Setting $u=0$ in equation (2.6), we have

$$\begin{aligned} {}_rp_x &= (p_x)^r \\ \frac{l_{x+r}}{l_x} &= \left(\frac{l_{x+1}}{l_x}\right)^r \\ \ln(l_{x+r}) - \ln(l_x) &= r\ln(l_{x+1}) - r\ln(l_x) \\ \ln(l_{x+r}) &= (1-r)\ln(l_x) + r\ln(l_{x+1}) \end{aligned}$$

□

You will find this equation – the interpolation between $\ln(l_x)$ and $\ln(l_{x+1})$ – useful when you are given a table of l_x.

Application of the Constant Force of Mortality Assumption to l_x

$$\ln(l_{x+r}) = (1-r)\ln(l_x) + r\ln(l_{x+1}), \quad \text{for } 0 \le r < 1$$

Example 2.6.

Assuming constant force of mortality between integral ages, repeat Example 2.4.

Solution:

(a) ${}_{0.26}p_{61} = (p_{61})^{0.26} = (99300/99700)^{0.26} = 0.998955.$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{61})$ and $\ln(l_{62})$ as follows: $\ln(l_{61.26}) = (1-0.26)\ln(l_{61}) + 0.26\ln(l_{62})$, which gives $l_{61.26} = 99595.84526$. Hence, ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99595.84526/99700 = 0.998955$.

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_{2}p_{60} \times {}_{0.2}p_{62} = 1 - {}_{2}p_{60} \times (p_{62})^{0.2}$

$$= 1 - \frac{l_{62}}{l_{60}}\left(\frac{l_{63}}{l_{62}}\right)^{0.2} = 1 - \frac{99300}{100000}\left(\frac{98800}{99300}\right)^{0.2} = 0.008002.$$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{62})$ and $\ln(l_{63})$ as follows: $\ln(l_{62.2}) = (1-0.2)\ln(l_{62}) + 0.2\ln(l_{63})$, which gives $l_{62.2} = 99199.79798$. Hence, ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 0.008002$.

(c) First, we consider ${}_{0.3}p_{62.8}$:

$${}_{0.3}p_{62.8} = {}_{0.2}p_{62.8} \times {}_{0.1}p_{63} = (p_{62})^{0.2}(p_{63})^{0.1}.$$

Hence,

$${}_{0.3}q_{62.8} = 1 - (p_{62})^{0.2}(p_{63})^{0.1} = 1 - \left(\frac{l_{63}}{l_{62}}\right)^{0.2}\left(\frac{l_{64}}{l_{63}}\right)^{0.1}$$
$$= 1 - \left(\frac{98800}{99300}\right)^{0.2}\left(\frac{98200}{98800}\right)^{0.1} = 0.001617.$$

Alternatively, we can calculate the answer by an interpolation between $\ln(l_{62})$ and $\ln(l_{63})$ and another interpolation between $\ln(l_{63})$ and $\ln(l_{64})$.

First, $\ln(l_{62.8}) = (1-0.8)\ln(l_{62}) + 0.8\ln(l_{63})$, which gives $l_{62.8} = 98899.79818$.

Second, $\ln(l_{63.1}) = (1-0.1)\ln(l_{63}) + 0.1\ln(l_{64})$, which gives $l_{63.1} = 98739.8354$.

Finally, ${}_{0.3}q_{62.8} = 1 - l_{63.1}/l_{62.8} = 0.001617$.

□

Example 2.7.

You are given the following life table:

x	l_x	d_x
90	1000	50
91	950	50
92	900	60
93	840	c_1
94	c_2	70
95	700	80

(a) Find the values of c_1 and c_2.

(b) Calculate ${}_{1.4}p_{90}$, assuming uniform distribution of deaths between integer ages.

(c) Repeat (b) by assuming constant force of mortality between integer ages.

Solution:

(a) We have $840 - c_1 = c_2$ and $c_2 - 70 = 700$. This gives $c_2 = 770$ and $c_1 = 70$.

(b) Assuming uniform distribution of deaths between integer ages, we have

$$\begin{aligned} {}_{1.4}p_{90} &= p_{90} \times {}_{0.4}p_{91} \\ &= p_{90}\,(1 - 0.4q_{91}) \\ &= \frac{l_{91}}{l_{90}}\left(1 - 0.4\frac{d_{91}}{l_{91}}\right) \\ &= \frac{950}{1000}\left(1 - 0.4 \times \frac{50}{950}\right) \\ &= 0.93. \end{aligned}$$

Alternatively, you can compute the answer by interpolating between l_{92} and l_{91}:

$$ {}_{1.4}p_{90} = p_{90} \times {}_{0.4}p_{91} = \frac{l_{91}}{l_{90}}\left(\frac{0.4l_{92} + 0.6l_{91}}{l_{91}}\right) = \frac{0.4 \times 900 + 0.6 \times 950}{1000} = 0.93 $$

(c) Assuming constant force of mortality between integer ages, we have

$$\begin{aligned} {}_{1.4}p_{90} &= p_{90} \times_{0.4} p_{91} \\ &= p_{90} \times (p_{91})^{0.4} \\ &= \frac{950}{1000}\left(\frac{900}{950}\right)^{0.4} \\ &= 0.92968. \end{aligned}$$

□

Let us conclude this section with the following table, which summarizes the formulas for the two fractional age assumptions.

	UDD	**Constant force**
${}_rp_x$	$1 - rq_x$	$(p_x)^r$
${}_rq_x$	rq_x	$1 - (p_x)^r$
μ_{x+r}	$\dfrac{q_x}{1 - rq_x}$	$-\ln(p_x)$

In the table, x is an integer and $0 \le r < 1$. The shaded formulas are the key formulas that you must remember for the examination.

2.3 Select-and-Ultimate Tables

Insurance companies typically assess risk before they agree to insure you. They cannot stay in business if they sell life insurance to someone who has just discovered he has only a few months to live. A team of underwriters will usually review information about you before you are sold insurance (although there are special insurance types called "guaranteed issue" which cannot be underwritten). For this reason, a person who has just purchased life insurance has a lower probability of death than a person the same age in the general population. The probability of death for a person who has just been issued life insurance is called a select probability. In this section, we focus on the modeling of select probabilities.

Let us define the following notation.

- $[x]$ indicates the age at selection (i.e., the age at which the underwriting was done).
- $[x]+t$ indicates a person currently age $x+t$ who was selected at age x (i.e., the underwriting was done at age x). This implies that the insurance contract has elapsed for t years.

For example, we have the following select probabilities:

- $q_{[x]}$ is the probability that a life age x now dies before age $x + 1$, given that the life is selected at age x.
- $q_{[x]+t}$ is the probability that a life age $x + t$ now dies before age $x + t + 1$, given that the life was selected at age x.

Due to the effect of underwriting, a select death probability $q_{[x]+t}$ must be no greater than the corresponding ordinary death probability q_{x+t}. However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the select period is n years, we have

$$\begin{aligned} q_{[x]+t} &< q_{x+t}, \quad \text{for } t < n. \\ q_{[x]+t} &= q_{x+t}, \quad \text{for } t \geq n. \end{aligned}$$

The ordinary death probability q_{x+t} is called the ultimate death probability. A life table that contains both select probabilities and ultimate probabilities is called a select-and-ultimate life table. The following is an excerpt of a (hypothetical) select-and-ultimate table with a select period of two years.

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
40	0.04	0.06	0.08	42
41	0.05	0.07	0.09	43
42	0.06	0.08	0.10	44
43	0.07	0.09	0.11	45

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. The death probabilities for this person are as follows:

Age 41: $q_{[41]} = 0.05$

Age 42: $q_{[41]+1} = 0.07$

Age 43: $q_{[41]+2} = q_{43} = 0.09$

Age 44: $q_{[41]+3} = q_{44} = 0.10$

Age 45: $q_{[41]+4} = q_{45} = 0.11$

As you see, the select-and-ultimate table is not difficult to use. We progress horizontally until we reach the ultimate death probability, then we progress vertically as when we are using an ordinary life table. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

Age 41: $q_{[40]+1} = 0.06$

Age 42: $q_{[40]+2} = q_{42} = 0.08$

Age 43: $q_{[40]+3} = q_{43} = 0.09$

Age 44: $q_{[40]+4} = q_{44} = 0.10$

Age 45: $q_{[40]+5} = q_{45} = 0.11$

Even though the two persons we considered are of the same age now, their current death probabilities are different. Because the first individual has the underwriting done more recently, the effect of underwriting on him/her is stronger, which means he/she should have a lower death probability than the second individual.

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x,k) = 1 - \frac{q_{[x]+k}}{q_{x+k}}.$$

For example, on the basis of the preceding table, $I(41{,}1) = 1 - q_{[41]+1}/q_{42} = 1 - 0.07/0.08 = 0.125$. If the effect of underwriting is strong, then $q_{[x]+k}$ would be small compared to q_{x+k}, and therefore $I(x,k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{[x]+k}$ would be close to q_{x+k}, and therefore $I(x,k)$ would be close to zero.

Let us first go through the following example, which involves a table of $q_{[x]}$.

Example 2.8. **[Course 3 Fall 2001 #2]**

For a select-and-ultimate mortality table with a 3-year select period:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.

(iii) White's age on 01/01/2001 is 61.

(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

(A) $0 \le P < 0.43$

(B) $0.43 \le P < 0.45$

(C) $0.45 \le P < 0.47$

(D) $0.47 \le P < 0.49$

(E) $0.49 \le P < 1.00$

Solution: White is now age 61 and was selected at age 60. So the probability that White will be alive 5 years from now can be expressed as $P = {}_5p_{[60]+1}$. We have

$$\begin{aligned} P &= {}_5p_{[60]+1} \\ &= p_{[60]+1} \times p_{[60]+2} \times p_{[60]+3} \times p_{[60]+4} \times p_{[60]+5} \\ &= p_{[60]+1} \times p_{[60]+2} \times p_{63} \times p_{64} \times p_{65} \\ &= (1 - q_{[60]+1})(1 - q_{[60]+2})(1 - q_{63})(1 - q_{64})(1 - q_{65}) \\ &= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17) \\ &= 0.4589. \end{aligned}$$

Hence, the answer is (C).

□

In some exam questions, a select-and-ultimate table may be used to model a real life problem. Take a look at the following example.

Example 2.9. **[MLC Spring 2012 #13]**

Lorie's Lorries rents lavender limousines.

On January 1 of each year they purchase 30 limousines for their existing fleet; of these, 20 are new and 10 are one-year old.

Vehicles are retired according to the following 2-year select-and-ultimate table, where selection is age at purchase:

Limousine age (x)	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
0	0.100	0.167	0.333	2
1	0.100	0.333	0.500	3
2	0.150	0.400	1.000	4
3	0.250	0.750	1.000	5
4	0.500	1.000	1.000	6
5	1.000	1.000	1.000	7

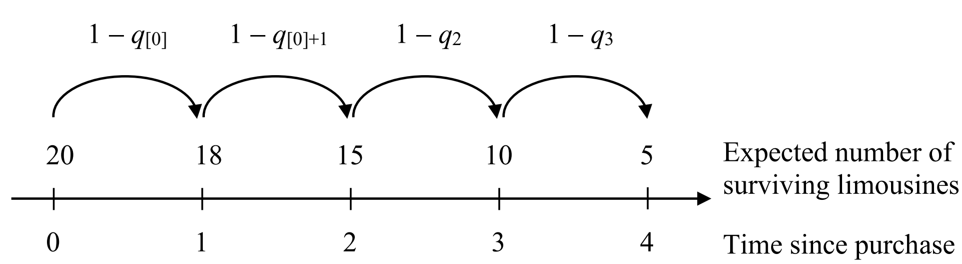
Lorie's Lorries has rented lavender limousines for the past ten years and has always purchased its limousines on the above schedule.

Calculate the expected number of limousines in the Lorie's Lorries fleet immediately after the purchase of this year's limousines.

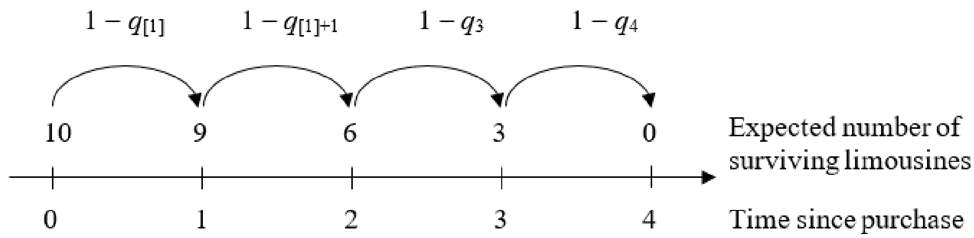
(A) 93 (B) 94 (C) 95 (D) 96 (E) 97

Solution: Let us consider a purchase of 30 limousines in a given year. According to information given, 20 of them are brand new while 10 of them are 1-year-old.

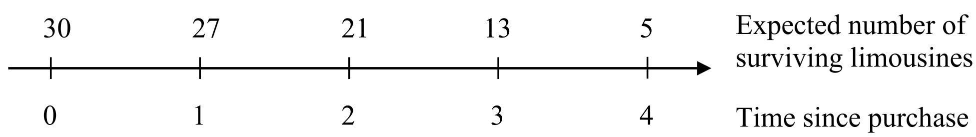
For the 20 brand new limousines, their "age at selection" is 0. As such, the sequence of "death" probabilities applicable to these 20 new limousines are $q_{[0]}$, $q_{[0]+1}$, q_2, q_3, q_4, q_5, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most four years since the time of purchase. For these 20 brand new limousines, the expected number of "survivors" limousines in each future year can be calculated as follows:



For the 10 1-year-old limousines, their "age at selection" is 1. As such, the sequence of "death" probabilities applicable to these 10 1-year-old limousines are $q_{[1]}$, $q_{[1]+1}$, q_3, q_4, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most three years since the time of purchase. For these 10 1-year-old limousines, the expected number of "surviving" limousines in each future year can be calculated as follows:



Considering the entire purchase of 30 limousines, we have the following:



Suppose that today is January 1, 2013. Since a limousine cannot last for more than four years since the time of purchase, the oldest limousine in Lorie's fleet should be purchased on January 1, 2009. Using the results above, the expected number of limousines on January 1, 2013 can be calculated as follows:

	1/1/09	1/1/10	1/1/11	1/1/12	1/1/13
Purchased on Jan 1, 2009	30	27	21	13	5
Purchased on Jan 1, 2010		30	27	21	13
Purchased on Jan 1, 2011			30	27	21
Purchased on Jan 1, 2012				30	27
Purchased on Jan 1, 2013					30

The answer is thus $5 + 13 + 21 + 27 + 30 = 96$, which corresponds to option (D).

□

Sometimes, you may be given a select-and-ultimate table that contains the life table function l_x. In this case, you can calculate survival and death probabilities by using the following equations:

$$ {}_sp_{[x]+t} = \frac{l_{[x]+t+s}}{l_{[x]+t}}, \quad {}_sq_{[x]+t} = 1 - \frac{l_{[x]+t+s}}{l_{[x]+t}}. $$

Let us study the following two examples.

Example 2.10. You are given the following select-and-ultimate table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	9907	9905	9901	32
31	9903	9900	9897	33
32	9899	9896	9892	34
33	9894	9891	9887	35

Calculate the following:

(a) ${}_2q_{[31]}$

(b) ${}_2p_{[30]+1}$

(c) ${}_{1|2}q_{[31]+1}$

Solution:

(a) $${}_2q_{[31]} = 1 - \frac{l_{[31]+2}}{l_{[31]}} = 1 - \frac{l_{33}}{l_{[31]}} = 1 - \frac{9897}{9903} = 0.000606.$$

(b) $${}_2p_{[30]+1} = \frac{l_{[30]+1+2}}{l_{[30]+1}} = \frac{l_{33}}{l_{[30]+1}} = \frac{9897}{9905} = 0.999192.$$

(c) $${}_{1|2}q_{[31]+1} = \frac{l_{[31]+1+1} - l_{[31]+1+1+2}}{l_{[31]+1}} = \frac{l_{33} - l_{35}}{l_{[31]+1}} = \frac{9897 - 9887}{9900} = 0.001010.$$ □

Exam questions such as the following may involve both $q_{[x]}$ and $l_{[x]}$.

Example 2.11. **[MLC Spring 2012 #1]**

For a 2-year select and ultimate mortality model, you are given:

(i) $q_{[x]+1} = 0.95q_{x+1}$

(ii) $l_{76} = 98{,}153$

(iii) $l_{77} = 96{,}124$

Calculate $l_{[75]+1}$.

(A) 96,150 (B) 96,780 (C) 97,420 (D) 98,050 (E) 98,690

Solution: From (ii) and (iii), we know that $q_{76} = 1 - 96124/98153 = 0.020672$.

From (i), we know that $q_{[75]+1} = 0.95q_{76} = 0.95 \times 0.020672 = 0.019638$.

Since

$$l_{[75]+2} = l_{[75]+1}(1 - q_{[75]+1}),$$

and $l_{[75]+2} = l_{77}$, we have $l_{[75]+1} = 96124/(1 - 0.019638) = 98049.5$. The answer is (D). □

Mock Test 1

****BEGINNING OF EXAMINATION****

1. For a whole life insurance of 1,000 issued to life selected at age x,

   (i) Percent of premium expenses is 90% in the first year, and 10% in each year thereafter.

   (ii) Maintenance expenses are 15 per 1,000 of insurance in the first year, and 3 per 1,000 of insurance thereafter.

   (iii) Claim settlement expenses are 10 per 1,000 of insurance.

   (iv) A 15-year select and ultimate mortality is to be used.

   Determine the expression of the gross premium for the policy using equivalence principle.

   (A) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (B) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

   (C) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{\ddot{a}_{[x]} - 0.9}$

   (D) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (E) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

2. The following results were obtained from a survival study, using the Kaplan-Meier (KM) estimator:

Time of death t in the sample	KM estimate of $S(t)$	Standard error of estimate
17	0.957	0.0149
25	0.888	0.0236
32	0.814	0.0298
36	0.777	0.0321
39	0.729	0.0348
42	0.680	0.0370
44	0.659	0.0378
47	0.558	0.0418
50	0.360	0.0470
54	0.293	0.0456
56	0.244	0.0440
57	0.187	0.0420
59	0.156	0.0404
62	0.052	0.0444

   Find the upper limit of the 95% log-confidence interval for $S(45)$.

   (A) 0.70 (B) 0.71 (C) 0.72 (D) 0.73 (E) 0.74

3. You are given:

(i) $i = 0.07$

(ii) $\ddot{a}_x = 11.7089$

(iii) $\ddot{a}_{x:\overline{40}|} = 11.55$

(iv) $\ddot{a}_{x+40} = 7.1889$

Find $A^{\ \ 1}_{x:\overline{40}|}$.

(A) 0.120 (B) 0.147 (C) 0.172 (D) 0.197 (E) 0.222

4. You are given the following select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.11	0.13	0.15	67
66	0.12	0.135	0.16	68
67	0.13	0.145	0.17	69

Deaths are uniformly distributed over each year of age.

Find ${}_{1.8}p_{[66]+0.6}$.

(A) 0.69 (B) 0.71 (C) 0.73 (D) 0.75 (E) 0.77

5. For a 3-year fully discrete endowment insurance of 1,000 on (x), you are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.15$

(iii) $v = 0.9$

(iv) Deaths are uniformly distributed over each year of age.

Calculate the net premium policy value 9 months after the issuance of the policy.

(A) 274 (B) 280 (C) 283 (D) 286 (E) 290

6. Which of the following statements is/are correct?

I. Insurable interest in an entity exists if one would suffer a financial loss if that entity is damaged.

II. Insurable interest is related to the concept of adverse selection.

III. Stranger owned life insurance is illegal in many jurisdiction because the purchaser has no insurable interest in the insured.

(A) I only (B) II only (C) I and III only

(D) II and III only (E) I, II and III

7. Which of the following is/are strictly increasing function(s) of T_x for all $T_x \geq 0$?

   I. The present value random variable for a continuous whole life annuity of \$1 on (x)

   II. The present value random variable for a continuous n-year temporary life annuity of \$1 on (x)

   III. The net future loss at issue random variable for a fully continuous whole life insurance of \$1 on (x)

   (A) I only (B) III only (C) I, II only
   (D) I, III only (E) I, II and III

8. For a fully discrete whole life insurance on (30), you are given:

   (i) $i = 0.05$

   (ii) $q_{29+h} = 0.004$

   (iii) The net amount at risk for policy year h is 1295.

   (iv) The terminal policy value for policy year $h - 1$ is 179.

   (v) $\ddot{a}_{30} = 16.2$

   Calculate the initial policy value for policy year $h + 1$.

   (A) 188 (B) 192 (C) 200 (D) 214 (E) 226

9. You are given:

   (i) $\mu_{x+t} = \begin{cases} 0.02 & 0 \leq t < 1 \\ 0.07 & 1 \leq t < 2 \end{cases}$ (ii) $Y = \min(T_x, 2)$

   Calculate $\mathrm{E}(Y)$.

   (A) 1.88 (B) 1.90 (C) 1.92 (D) 1.94 (E) 1.96

10. An insurer issues fully discrete endowment insurance policies to a group of 50 high-risk drivers all aged 35 with a sum insured of 2,000. You are given:

   (i) The mortality of each driver follows the Standard Ultimate Life Table with an age rating of 3 years. That is,
   $$q_x = q_{x+3}^{SULT},$$
   where q_y^{SULT} is the 1-year death probability under the Standard Ultimate Life Table.

   (ii) Lifetimes of the group of 50 high-risk drivers are independent.

   (iii) Premiums are payable annually in advance. Each premium is 110% of the net annual premium.

   (iv) $i = 0.05$

   By assuming a normal approximation, estimate the 95$^{\text{th}}$-percentile of the net future loss random variable.

   (A) 1090 (B) 1240 (C) 1390 (D) 1540 (E) 1690

11. For a population, the force of mortality is $\mu_x = 0.01\pi \tan(0.01\pi x)$ for $0 < x < 50$.

Calculate ${}_{10|5}q_{20}$.

[Note: The anti-derivative of $\tan x$ is $-\ln|\cos x|$.]

(A) 0.15 (B) 0.17 (C) 0.19 (D) 0.21 (E) 0.23

12. Which of the following regarding is/are correct?

I. Variable annuity has cash value.

II. Term insurance has cash value.

III. Universal life insurance has cash value.

(A) I only (B) II only (C) III only

(D) II and III only (E) I, II and III

13. You are given the following about 100 insurance policies in a study of time to policy surrender:

(i) The study was designed in such a way that for every policy that was surrendered, a new policy was added, meaning that the risk set, r_j, is always equal to 100.

(ii) Policies are surrendered only at the end of a policy year.

(iii) The number of policies surrendered at the end of each policy year was observed to be:

$$\begin{gathered} 2 \text{ at the end of the } 1^{\text{st}} \text{ policy year} \\ 3 \text{ at the end of the } 2^{\text{nd}} \text{ policy year} \\ 4 \text{ at the end of the } 3^{\text{rd}} \text{ policy year} \\ \vdots \\ n+1 \text{ at the end of the } n^{\text{th}} \text{ policy year} \end{gathered}$$

(iv) The Nelson-Aalen empirical estimate of the cumulative distribution function at time n is 0.41725.

Find n.

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

14. You are given:

(i) $\delta = 0.05$

(ii) $\bar{A}_x = 0.44$

(iii) ${}^2\bar{A}_x = 0.22$

Consider a portfolio of 100 fully continuous whole life insurances. The ages of the all insureds are x, and their lifetimes are independent. The face amount of the policies, the premium rate and the number of policies are as follows:

Face amount	Premium rate	Number of Policies
100	4.3	75
400	17.5	25

By using a normal approximation, calculate the probability that the present value of the aggregate loss-at-issue for the insurer's portfolio will exceed 700.

(A) $1 - \Phi(2.28)$ (B) $1 - \Phi(0.17)$ (C) $\Phi(0)$
(D) $\Phi(0.17)$ (E) $\Phi(2.28)$

15. Victor is now age 22, and his future lifetime has the following cumulative distribution function:

$$\mathrm{F}_{22}(t) = 1 - (1 + 0.04t)e^{-0.04t}.$$

Let Z be the present value random variable for a fully continuous life insurance that pays 100 immediately on the death of Victor provided that he dies between ages 32 and 52.

The force of interest is 0.06.

Find the 70^{th}-percentile of Z.

(A) 0 (B) 25 (C) 40 (D) 47 (E) 50

16. You are given:

(i) $i = 0.05$

(ii) $A_{60} = 0.560$

(iii) $A_{40} = 0.176$

(iv) ${}_{20}p_{40} = 0.75$

Calculate $a^{(3)}_{40:\overline{20|}}$ using the two-term Woolhouse's approximation.

(A) 13.7 (B) 14.2 (C) 14.5 (D) 14.9 (E) 15.1

17. You are given:

(i) $\delta = 0.05$

(ii) $\ddot{a}_{60} = 12.18$

(iii) $p_{60} = 0.98$

Using the claims accelerated approach, calculate $\ddot{a}_{61}^{(6)}$.

(A) 11.6 (B) 11.7 (C) 11.8 (D) 11.9 (E) 12.1

18. Let Y be the present value random variable for a special three-year temporary life annuity on (x). You are given:

(i) The life annuity pays $2 + k$ at time k, for $k = 0$, 1 and 2.

(ii) $v = 0.9$

(iii) $p_x = 0.8$, $p_{x+1} = 0.75$, $p_{x+2} = 0.5$

Calculate the standard deviation of Y.

(A) 1.2 (B) 1.8 (C) 2.4 (D) 3.0 (E) 3.6

19. You are given:

$$\mu_x = \begin{cases} 0.04 & 50 \le x < 60 \\ 0.05 + 0.001(x-60)^2 & 60 \le x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38 (B) 0.44 (C) 0.47 (D) 0.50 (E) 0.56

20. You are given:

(i) $p_{41} = 0.999422$

(ii) $i = 0.03$

(iii) $\ddot{a}_{42:\overline{23}|} = 16.7147$

Calculate the Full Preliminary Term reserve at time 2 for a 25-year fully discrete endowment insurance, issued to (40), with sum insured 75,000.

(A) 2,188 (B) 2,190 (C) 2,192 (D) 2,194 (E) 2,196

** END OF EXAMINATION **

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1. [Chapter 7] Answer: (B)

   Let G be the gross premium. By the equivalence principle,

   APV of gross premiums = APV of death benefit + APV of expenses.

$$\begin{aligned} G\ddot{a}_{[x]} &= 1010\bar{A}_{[x]} + 0.9G + 0.1Ga_{[x]} + 15 + 3a_{[x]} \\ &= 1010\bar{A}_{[x]} + 0.9G + 0.1G(\ddot{a}_{[x]} - 1) + 15 + 3(\ddot{a}_{[x]} - 1) \\ G &= \frac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8} \end{aligned}$$

2. [Chapter 8] Answer: (D)

   $U = \dfrac{1.96\text{SE}[\hat{S}(t)]}{\hat{S}(t)\ln\hat{S}(t)} = \dfrac{1.96 \times 0.0378}{0.659\ln 0.659} = -0.26958$, and the upper limit is

   $0.659^{\exp(-0.26958)} = 0.659^{0.763697} = 0.727249$.

3. [Chapter 4] Answer: (E)

   We first change all annuities into insurances.

   Statement (ii) implies that $A_x = 1 - \frac{0.07}{1.07} \times 11.7089 = 0.2340$.

   Statement (iii) implies that $A_{x:\overline{40}|} = 1 - \frac{0.07}{1.07} \times 11.55 = 0.2444$.

   Statement (iv) implies that $A_{x+40} = 1 - \frac{0.07}{1.07} \times 7.1889 = 0.5297$.

   Finally, by $A_x = A^1_{x:\overline{40}|} + A_{x:\overset{\;\,1}{\overline{40}|}}A_{x+40}$ and $A_{x:\overset{\;\,1}{\overline{40}|}} + A^1_{x:\overline{40}|} = A_{x:\overline{40}|}$,

$$0.2340 = A^1_{x:\overline{40}|} + (0.2444 - A^1_{x:\overline{40}|}) \times 0.5297$$

   On solving, we get $A^1_{x:\overline{40}|} = \dfrac{0.2340 - 0.2444 \times 0.5297}{1 - 0.5297} = 0.2223$.

4. [Chapter 2] Answer: (E)

   <u>Method 1:</u>
$$\begin{aligned} {}_{1.8}p_{[66]+0.6} &= {}_{0.4}p_{[66]+0.6} \times {}_{1.4}p_{[66]+1} \\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{[66]+2} \\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{68} \end{aligned}$$

   Obviously, $p_{[66]+1} = 1 - 0.135 = 0.865$, ${}_{0.4}p_{68} = 1 - 0.4q_{68} = 1 - 0.4(0.16) = 0.936$.

   Finally, ${}_{0.4}p_{[66]+0.6} = p_{[66]}/{}_{0.6}p_{[66]} = 0.88/(1 - 0.6 \times 0.12) = 0.948276$.

   So, the answer is ${}_{1.8}p_{[66]+0.6} = 0.76776$.

   <u>Method 2:</u> ${}_{1.8}p_{[66]+0.6} = l_{[66]+2.4}/l_{[66]+0.6}$

   Without loss of generality, let $l_{[66]} = 100$. Then we have:

   $l_{[66]+1} = 88$, $l_{[66]+2} = 88 \times 0.865 = 76.12$, $l_{[66]+2} = 76.12 \times 0.84 = 63.9408$.

   So, $l_{[66]+2.4} = 0.4 \times 63.9408 + 0.6 \times 76.12 = 71.24832$,

   $l_{[66]+0.4} = 0.6 \times 88 + 0.4 \times 100 = 92.8$, and ${}_{1.8}p_{[66]+0.6} = 0.767762$.

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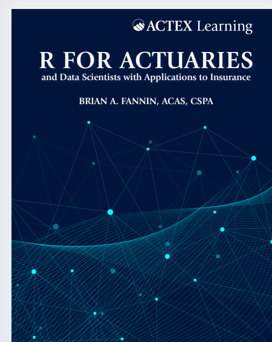
Brian Fannin, ACAS, CSPA



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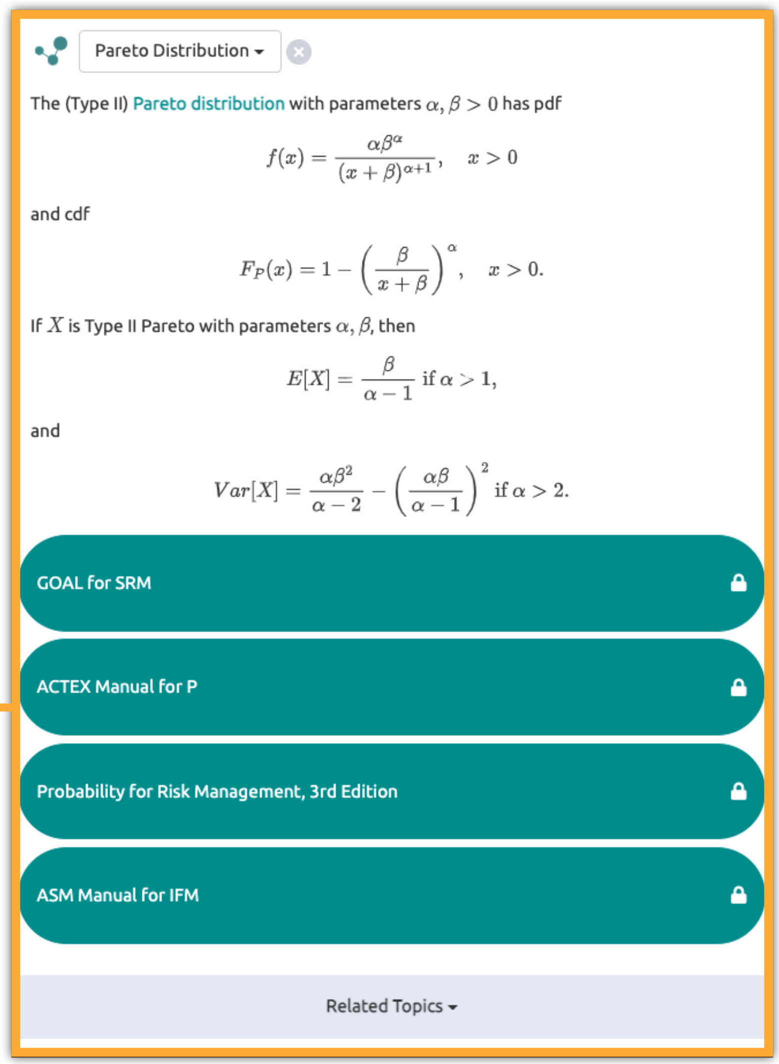

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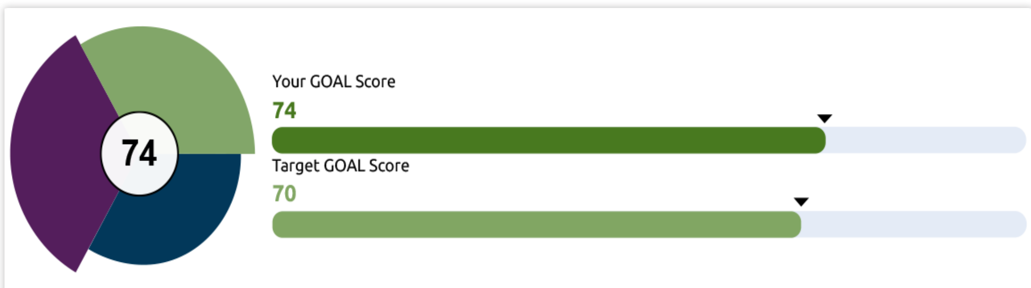

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See key areas where you can improve.

General Probability — Set functions including set notation and basic elements of probability

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Created	Last Accessed	Completed	Mode	Categories	Questions	Status	Status
05/24/2022 21:57:43	05/24/2022 21:57:43		Quiz	Continuous P...	25	New	Resume
05/24/2022 10:34:05	05/24/2022 14:57:49	05/24/2022 14:57:49	Practice Session	Addition and ...	80	Complete	Review
05/21/2022 19:32:50	05/23/2022 20:02:00		Simulated Exam	Exam 2	30	Reviewing	Complete
05/17/2022 15:19:19	05/17/2022 15:46:03	05/23/2022 14:15:29	Simulated Exam	Exam 6	30	Complete	Review
05/14/2022 11:26:59	05/14/2020 12:02:36	05/23/2022 11:57:47	Quiz	Conditional D...	20	Complete	Review

Quickly return to previous sessions.

Contents

INTRODUCTORY COMMENTS

The FAM exam is divided almost equally into FAM-S and FAM-L topics. This study guide is designed to help in the preparation for the Society of Actuaries FAM-S Exam.

The first part of this manual consists of a summary of notes, illustrative examples and problem sets with detailed solutions. The second part consists of 4 practice exams. The SOA exam syllabus for the FAM exam indicates that the exam is 3.5 hours in length with 40 multiple choice questions. The practice exams in this manual each have 20 questions, reflecting the fact that FAM-S is 50% of the full FAM exam. The appropriate time for the 20 question FAM-S practice exams in this manual is one hour and forty-five minutes.

The level of difficulty of the practice exam questions has been designed to be similar to those on past exams covering the same topics. The practice exam questions are not from old SOA exams.

I have attempted to be thorough in the coverage of the topics upon which the exam is based, and consistent with the notation and content of the official references. I have been, perhaps, more thorough than necessary on reviewing maximum likelihood estimation.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that you have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types. I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study guide is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

The notes and examples are divided into 32 sections of varying lengths, with some suggested time frames for covering the material. There are almost 180 examples in the notes and over 440 exercises in the problem sets, all with detailed solutions. The 4 practice exams have 20 questions each, also with detailed solutions. Some of the examples and exercises are taken from previous SOA exams. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions. In total there are almost 700 examples/problems/sample exam questions with detailed solutions. ACTEX gratefully acknowledges the SOA for allowing the use of their exam problems in this study guide.

I suggest that you work through the study guide by studying a section of notes and then attempting the exercises in the problem set that follows that section. The order of the sections of notes is the order that I recommend in covering the material, although the material on pricing and reserving in Sections 27 to 31 is independent of the other material on the exam. The order of topics in this manual is not the same as the order presented on the exam syllabus.

It has been my intention to make this study guide self-contained and comprehensive for the FAM-S Exam topics, however it is important to be familiar with original reference material on all topics.

While the ability to derive formulas used on the exam is usually not the focus of an exam question, it is useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There may be an occasional reference in the review notes to a derivation, but you are encouraged to review the official reference material for more detail on formula derivations.

In order for the review notes in this study guide to be most effective, you should have some background at the junior or senior college level in probability and statistics. It will be assumed that you are reasonably familiar with differential and integral calculus. The prerequisite concepts to modeling and model estimation are reviewed in this study guide. The study guide begins with a detailed review of probability distribution concepts such as distribution function, hazard rate, expectation and variance. Of the various calculators that are allowed for use on the exam, I am most familiar with the BA II PLUS. It has several easily accessible memories. The TI-30X IIS has the advantage of a multi-line display. Both have the functionality needed for the exam.

There is a set of tables that has been provided with the exam in past sittings. These tables consist of some detailed description of a number of probability distributions along with tables for the standard normal and chi-squared distributions. The tables can be downloaded from the SOA website www.soa.org.

If you have any questions, comments, criticisms or compliments regarding this study guide, please contact the publisher ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. ACTEX will be maintaining a website for errata that can be accessed from www.actexmadriver.com. It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

Samuel A. Broverman July 2022

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Section 2

Review of Random Variables I

Probability, Density and Distribution Functions

This section covers preliminary topics needed for the later FAM syllabus material and Section 2.7 relates to Section 4.2.4 in the "Loss Models" book. Suggested time frame for covering this section is two hours. A brief review of some basic calculus relationships is presented first.

2.1 Calculus Review

Natural logarithm and exponential functions

$\ln(x) = \log(x)$ is the logarithm to the base e ;

$$\begin{array}{lll} \ln(e) = 1 \ , & \ln(1) = 0 \ , & e^0 = 1 \ , \\ \ln(e^y) = y \ , & e^{\ln(x)} = x \ , & \ln(a^y) = y \times \ln(a) \ , \\ \ln(y \times z) = \ln(y) + \ln(z) \ , & \ln\left(\frac{y}{z}\right) = \ln(y) - \ln(z) \ , & e^x e^z = e^{x+z} \ , \\ (e^x)^w = e^{xw} & & \end{array} \tag{2.1}$$

Differentiation

For the function $f(x)$, $\quad f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (2.2)

Product rule: $\frac{d}{dx}[g(x) \times h(x)] = g'(x) \times h(x) + g(x) \times h'(x)$ (2.3)

Quotient rule: $\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{h(x) \times g'(x) - g(x) \times h'(x)}{[h(x)]^2}$ (2.4)

Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x) \ , \ \frac{d}{dx} \ln[g(x)] = \frac{g'(x)}{g(x)}$

$\frac{d}{dx}[g(x)]^n = n \times [g(x)]^{n-1} \times g'(x) \ , \ \frac{d}{dx} a^x = a^x \times \ln(a)$ (2.5)

Integration

$$\int \boldsymbol{x^n \ dx = \frac{x^{n+1}}{n+1} + c} \ , \ \int \boldsymbol{a^x \ dx = \frac{a^x}{\ln(a)} + c} \ , \ \int \boldsymbol{\frac{1}{a+bx} \ dx = \frac{1}{b}} \times \boldsymbol{\ln[a+bx] + c} \tag{2.6}$$

Integration by parts

for definite integrals: $\int_a^b u(t) \ dv(t) = u(b) \times v(b) - u(a) \times v(a) - \int_a^b v(t) \ du(t)$ (2.7)

for indefinite integrals: $\int u \ dv = uv - \int v \ du$

(this is derived by integrating both sides of the product rule).

Note that $dv(t) = v'(t)\ dt$ and $du(t) = u'(t)\ dt$.

Some additional relationships involving integration:

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x), \quad \frac{d}{dx}\int_x^b g(t)\,dt = -g(x) \tag{2.8}$$

$$\frac{d}{dx}\int_{h(x)}^{j(x)} g(t)dt = g(j(x)) \times j'(x) - g(h(x)) \times h'(x) \tag{2.9}$$

$$\int_0^\infty x^n e^{-kx}\,dx = \frac{n!}{k^{n+1}} \quad \text{if} \quad k > 0 \quad \text{and } n \text{ is an integer } \geq 0 \tag{2.10}$$

The word "model" used in the context of a loss model, usually refers to the distribution of a **loss random variable**. Random variables are the basic components used in actuarial modeling. In this section we review the definitions and illustrate the variety of random variables that we will encounter in the FAM Exam material.

A **random variable** is a numerical quantity that is related to the outcome of some random experiment on a probability space. For the most part, the random variables we will encounter are the numerical outcomes of some loss related event such as the dollar amount of claims in one year from an auto insurance policy, or the number of tornados that touch down in Kansas in a one year period.

2.2 Discrete Random Variable

The random variable X is **discrete** and is said to have a **discrete distribution** if it can take on values only from a finite or countably infinite sequence (usually the integers or some subset of the integers). As an example, consider the following two random variables related to successive tosses of a coin:

$X = 1$ if the first head occurs on an even-numbered toss, $X = 0$ if the first head occurs on an odd-numbered toss;

$Y = n$, where n is the number of the toss on which the first head occurs.

Both X and Y are discrete random variables, where X can take on only the values 0 or 1, and Y can take on any positive integer value.

Probability function of a discrete random variable

The **probability function (pf)** of a discrete random variable is usually denoted $p(x)$ (or $f(x)$), and is equal to $P[X = x]$. As its name suggests, the probability function describes the probability of individual outcomes occurring.

The probability function must satisfy the following two conditions:

$$(i)\ 0 \leq p(x) \leq 1 \quad \text{for all } x, \qquad \text{and} \qquad (ii)\ \sum_{all\ x} p(x) = 1 \tag{2.11}$$

For the random variable X above, the probability function is $p(0) = \dfrac{2}{3}$, $p(1) = \dfrac{1}{3}$,

and for Y it is $p(k) = \dfrac{1}{2^k}$ for $k = 1,2,3,\ldots$.

An event A is a subset of the set of all possible outcomes of X, and the probability of event A occurring is $P[A] = \displaystyle\sum_{x \in A} p(x)$.

For Y above, $P[Y \text{ is even}] = P[Y = 2,4,6,...] = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1}{3}$,

and this is also equal to $P[X = 1]$.

Some specific discrete random variables will be considered in some detail a little later in this manual, but the following is a brief description of a few important discrete distributions..

Discrete uniform distribution
If N is an integer ≥ 1, the **discrete uniform distribution** on the integers from 1 to N has probability function $P[X = k] = p(k) = \frac{1}{N}$ for $k = 1, \cdots N$ and $p(k) = 0$ otherwise. The discrete uniform distribution with $N = 6$ would apply to the outcome of the toss of a fair die.

Binomial distribution
The **binomial distribution** with parameters m (integer ≥ 1) and number q $(0 < q < 1)$ has probability function $P[X = k] = p(k) = \binom{m}{k} q^k (1-q)^{m-k}, k = 0,1,...,m$ where $\binom{m}{k} = \frac{m!}{k! \times (m-k)!}$ is the "binomial coefficient", and $p(k) = 0$ otherwise. The binomial distribution describes the number of "successful outcomes" out of m trials of a random "experiment" in which trials are mutually independent and each trial results in a successful outcome with probability q or unsuccessful outcome with probability $1 - q$.

Poisson distribution
The **Poisson distribution** with parameter λ has probability function
$P[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k \geq 0$, where k is an integer. The Poisson distribution is a very important distribution in actuarial applications to the modeling of the number of events occurring in a specified period of time.

2.3 Continuous Random Variable

A **continuous random variable** usually can assume numerical values from an interval of real numbers, perhaps the entire set of real numbers. As an example, the length of time between successive streetcar arrivals at a particular (in service) streetcar stop could be regarded as a continuous random variable (assuming that time measurement can be made perfectly accurate).

Probability density function
A continuous random variable X has a **probability density function (pdf)** denoted $f(x)$ or $f_X(x)$ (or sometimes denoted $p(x)$), which is a continuous function (except possibly at a finite or countably infinite number of points). For a continuous random variable, we do not describe probability at single points. We describe probability in terms of intervals. In the streetcar example, we would not define the probability that the next street car will arrive in exactly 1.23 minutes, but rather we would define a probability such as the probability that the streetcar will arrive between 1 and 1.5 minutes from now.

Probabilities related to X are found by integrating the density function over an interval.

$P[X \in (a,b)] = P[a < X < b]$ is defined to be equal to $\int_a^b f(x)\, dx$.

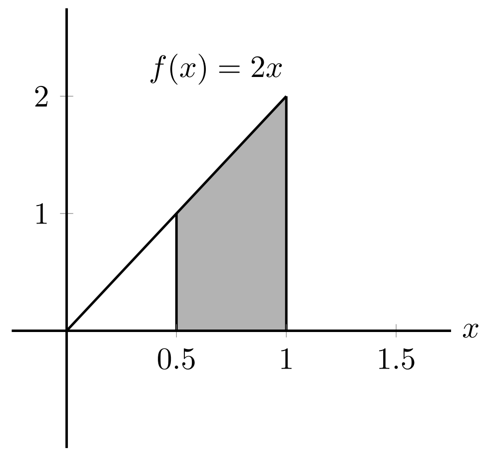
A pdf $f(x)$ must satisfy (i) $f(x) \geq 0$ for all x and (ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$ (2.12)

Often, the region of non-zero density is a finite interval, and $f(x) = 0$ outside that interval. If $f(x)$ is continuous except at a finite number of points, then probabilities are defined and calculated as if $f(x)$ was continuous everywhere (the discontinuities are ignored).

For example, suppose that X has density function $f(x) = \begin{cases} 2x & \text{for} \quad 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$.

Then f satisfies the requirements for a density function, since $\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 2x\, dx = 1$.

Then, for example $P[.5 < X < 1] = \int_{0.5}^{1} 2x\, dx = x^2\Big|_{0.5}^{1} = 0.75$. This is illustrated in the shaded area in the graph below.



For a continuous random variable X,

$$P[a < X < b] = P[a \leq X < b] = P[a < X \leq b] = P[a \leq X \leq b],$$

so that when calculating the probability for a continuous random variable on an interval, it is irrelevant whether or not the endpoints are included. For a continuous random variable, $P[X = a] = 0$ for any point a; non-zero probabilities only exist over an interval, not at a single point.

Some specific continuous distributions will be considered in some detail later in this manual, but the following is a brief description of a few important continuous distributions.

Continuous uniform distribution

If a and b are real numbers with $a < b$, the **continuous uniform distribution** on the interval (a,b) has pdf $f(x) = \frac{1}{b-a}$ for $a < x < b$, and $f(x) = 0$, otherwise.

Exponential distribution

If $\lambda > 0$ is a real number, then the **exponential distribution** with mean λ has pdf $f(x) = \frac{e^{-\lambda x}}{\lambda}$ for $x > 0$, and $f(x) = 0$, otherwise. The exponential distribution and generalizations of it are very important in actuarial modelling.

Another very important distribution, central to probability and statistics, is the **normal distribution**. This distribution will be considered in some detail a little later in this section.

2.4 Mixed Distribution

A random variable may have some points with non-zero **probability mass** and with a continuous pdf elsewhere. Such a distribution may be referred to as a **mixed distribution**, but the more general notion of mixtures of distributions will be covered later. The sum of the probabilities at the discrete points of probability plus the integral of the density function on the continuous region for X must be 1. For example, suppose that X has probability of 0.5 at $X = 0$, and X is a continuous random variable on the interval (0,1) with density function $f(x) = x$ for $0 < x < 1$, and X has no density or probability elsewhere. This satisfies the requirements for a random variable since the total probability is

$$P[X = 0] + \int_0^1 f(x)\, dx = 0.5 + \int_0^1 x\, dx = 0.5 + 0.5 = 1.$$

Then,

$$P[0 < X < 0.5] = \int_0^{.5} x\, dx = 0.125,$$

and

$$P[0 \le X < 0.5] = P[X = 0] + P[0 < X < 0.5] = 0.5 + 0.125 = 0.625.$$

Notice that for this random variable $P[0 < X < 0.5] \ne P[0 \le X < 0.5]$ because there is a probability mass at $X = 0$.

2.5 Cumulative Distribution, Survival and Hazard Functions

Given a random variable X, the **cumulative distribution function** of X (also called the **distribution function**, or cdf) is $F(x) = P[X \le x]$ (also denoted $F_X(x)$).

The cdf $F(x)$ is the "left-tail" probability, or the probability to the left of and including x.

The **survival function** is the complement of the distribution function,

$$S(x) = 1 - F(x) = P[X > x]. \tag{2.13}$$

The event $X > x$ is referred to as a "tail" or right tail of the distribution.

For any cdf $P[a < X \le b] = F(b) - F(a), \quad \lim_{x\to\infty} F(x) = 1, \quad \lim_{x\to-\infty} F(x) = 0. \qquad (2.14)$

For a discrete random variable with probability function $p(x)$, $F(x) = \sum_{w \le x} p(w)$, and in this case $F(x)$ is a "step function" (see Example 2.1 below); it has a jump (or step increase) at each point that has non-zero probability, while remaining constant until the next jump. Note that for a discrete random variable, $F(x)$ includes the probability at the point x as well as the sum of the probabilities of all the points to the left of x.

If X has a continuous distribution with density function $f(x)$, then

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt \quad \text{and} \quad S(x) = P(X > x) = \int_x^{\infty} f(t)\, dt \tag{2.15}$$

and $F(x)$ is a continuous, differentiable, non-decreasing function such that

$$\frac{d}{dx} F(x) = F'(x) = -S'(x) = f(x).$$

Also, for a continuous random variable, the **hazard rate** or **failure rate** is

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} = -\frac{d}{dx} \ln S(x). \tag{2.16}$$

The **cumulative hazard function** is

$$H(x) = \int_0^x h(t)\ dt \tag{2.17}$$

If X is continuous and $X \geq 0$, then the survival function satisfies

$$S(0) = 1 \quad \text{and} \quad S(x) = e^{-\int_0^x h(t)\ dt} = e^{-H(x)}.$$

If X has a mixed distribution with some discrete points and some continuous regions, then $F(x)$ is continuous except at the points of non-zero probability mass, where $F(x)$ will have a jump.

The region of positive probability of a random variable is called the **support** of the random variable.

2.6 Examples of Distribution Functions

The following examples illustrate the variety of distribution functions that can arise from random variables. The **support of a random variable** is the set of points over which there is positive probability or density.

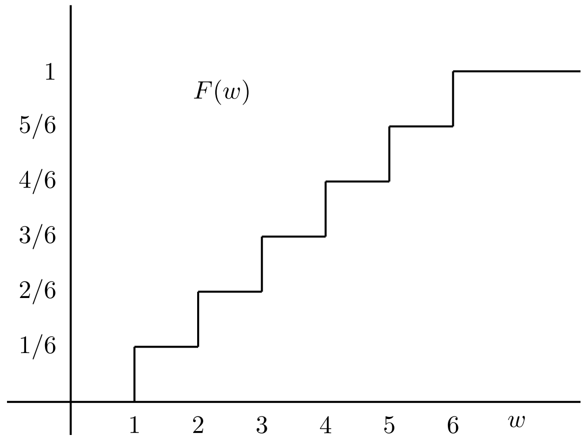
Example 2.1. Finite Discrete Random Variable (finite support)

W = number turning up when tossing one fair die. W has probability function

$p_W(w) = P[W = w] = \dfrac{1}{6}$ for w = 1,2,3,4,5,6

$$F_W(w) = P[W \leq w] = \begin{cases} 0 & \text{if} \quad w < 1 \\ \frac{1}{6} & \text{if} \quad 1 \leq w < 2 \\ \frac{2}{6} & \text{if} \quad 2 \leq w < 3 \\ \frac{3}{6} & \text{if} \quad 3 \leq w < 4 \\ \frac{4}{6} & \text{if} \quad 4 \leq w < 5 \\ \frac{5}{6} & \text{if} \quad 5 \leq w < 6 \\ 1 & \text{if} \quad w \geq 6 \end{cases}$$

The graph of the cdf is a step-function that increases at each point of probability by the amount of probability at that point (all 6 points have probability $\frac{1}{6}$ in this example). Since the support of W is finite (the support is the set of integers from 1 to 6), $F_W(w)$ reaches 1 at the largest point $W = 6$ (and stays at 1 for all $w \geq 6$).



□

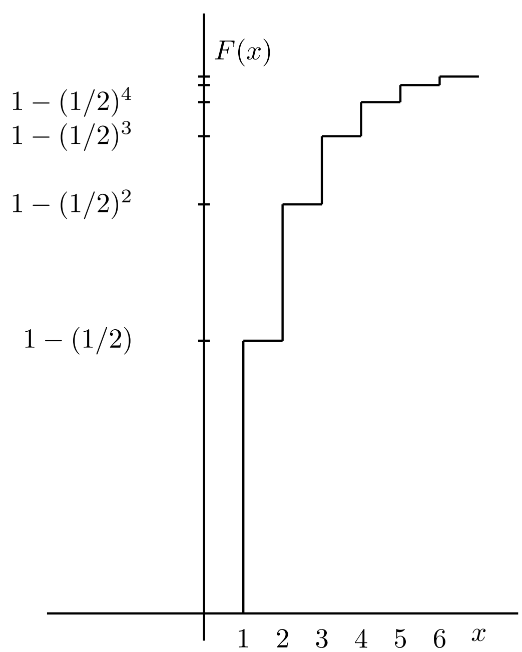
Example 2.2. Infinite Discrete Random Variable (infinite support)

X = the toss number of successive independent tosses of a fair coin on which the first head turns up.

X can be any integer ≥ 1, and the probability function of X is $p_X(x) = \dfrac{1}{2^x}$.

The cdf is $F_X(x) = \sum_{k=1}^{x} \dfrac{1}{2^k} = 1 - \dfrac{1}{2^x}$ for $x = 1,2,3,\ldots$.

The graph of the cdf is a step-function that increases at each point of probability by the amount of probability at that point. Since the support of X is infinite (the support is the set of integers ≥ 1) $F_X(x)$ never reaches 1, but approaches 1 as a limit as $x \to \infty$. The graph of $F_X(x)$ is



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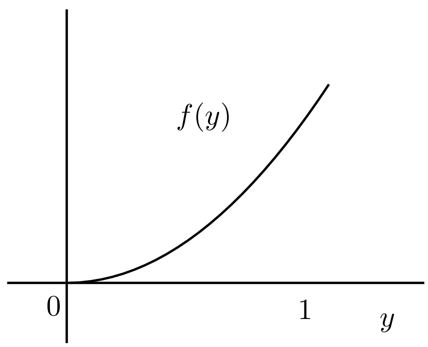
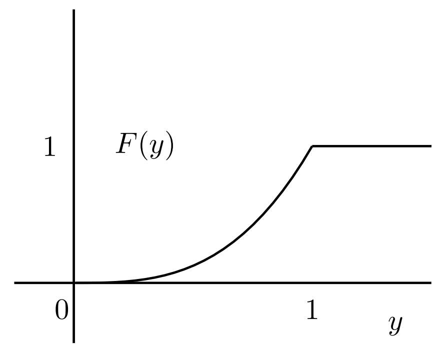
Example 2.3.

Continuous Random Variable on a Finite Interval

Y is a continuous random variable on the interval (0,1) with density function

$$f_Y(y) = \begin{cases} 3y^2 & \text{for} \quad 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}. \quad \text{Then} \quad F_Y(y) = \begin{cases} 0 & \text{if} \quad y < 0 \\ y^3 & \text{if} \quad 0 \le y < 1 \\ 1 & \text{if} \quad y \ge 1 \end{cases}.$$





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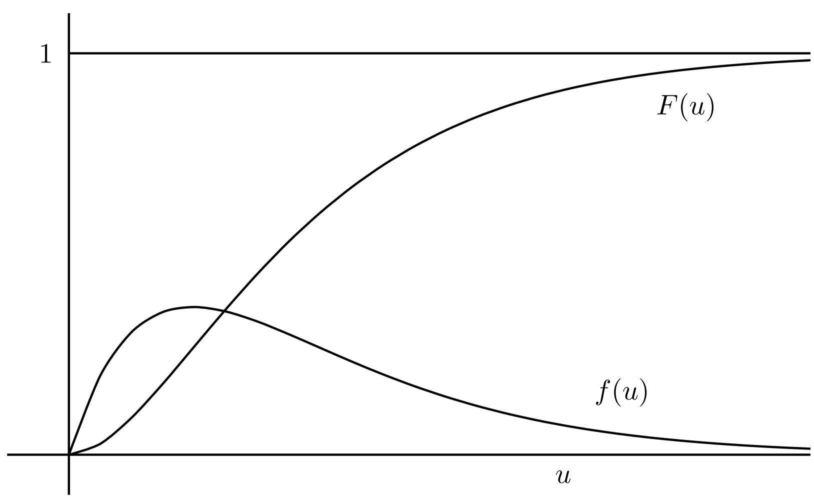
Example 2.4.

Continuous Random Variable on an Infinite Interval

U is a continuous random variable on the interval $(0,\infty)$ with density function

$$f_U(u) = \begin{cases} ue^{-u} & \text{for} \quad u > 0 \\ 0 & \text{for} \quad u \le 0 \end{cases}. \quad \text{Then} \quad F_U(u) = \begin{cases} 0 & \text{for} \quad u \le 0 \\ 1 - (1+u)e^{-u} & \text{for} \quad u > 0 \end{cases}.$$



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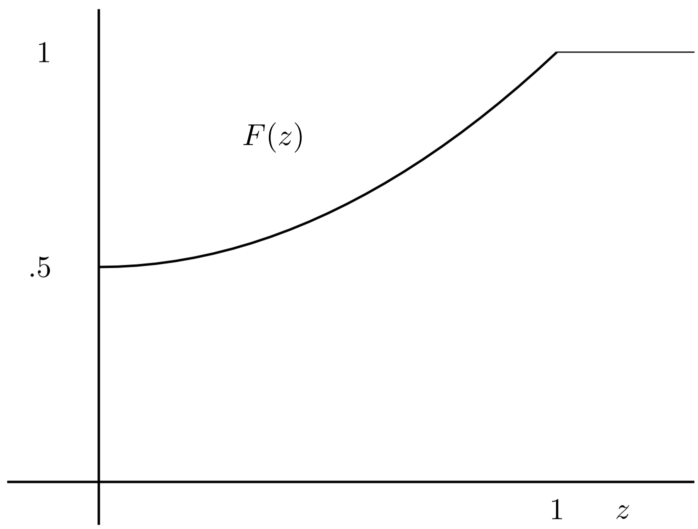
Example 2.5. Mixed Random Variable

Z has a mixed distribution on the interval [0,1). Z has probability of 0.5 at $Z = 0$, and Z has density function $f_Z(z) = z$ for $0 < z < 1$, and Z has no density or probability elsewhere. Then,

$$F_Z(z) = \begin{cases} 0 & \text{if} \quad z < 0 \\ 0.5 & \text{if} \quad z = 0 \\ 0.5 + \frac{1}{2}z^2 & \text{if} \quad 0 < z < 1 \\ 1 & \text{if} \quad z \geq 1 \end{cases}$$



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2.7 The Empirical Distribution

The **empirical distribution** is a discrete random variable constructed from a random sample. Suppose that the random sample consists of n observations, say x_1,x_2,...,x_n. If the data is from a loss distribution, then the x_i's are loss amounts, and if the data is from a survival distribution, they are times of death or failure. Either way, knowing the exact value of each outcome is what is referred to as complete data.

The empirical distribution assigns a probability of $\frac{1}{n}$ to each x_j. There may be some repeated numerical values of the observations, so let us suppose that there are k distinct numerical values that have been observed (some possibly repeated). Let us assume that these k values have been ordered from smallest to largest as $y_1 < y_2 < \ldots < y_k$, with s_j =number of observations equal to y_j (so that $s_1 + s_2 + \cdots + s_k = n$, the total number of observed values).

For instance, if we have a sample of $n = 8$ points, say x_1,...,x_8 that are 7, 2, 4, 4, 6, 2, 1, 9, then we have $k = 6$ distinct values (in numerical order) $y_1 = 1$, $y_2 = 2, y_3 = 4, y_4 = 6, y_5 = 7$, $y_6 = 9$, with $s_1 = 1$, $s_2 = 2, s_3 = 2, s_4 = 1, s_5 = 1$, $s_6 = 1$. This empirical distribution is a 6-point discrete random variable based on the numerical values of the y's, and it has all the properties of a discrete random variable.

The **empirical distribution probability function** is defined to be

$$p_n(y_j) = \frac{\text{number of } x_i\text{'s that are equal to } y_j}{n} = \frac{s_j}{n} \tag{2.18}$$

(a probability of $\frac{1}{n}$ is assigned to each of the n observations, and s_j denotes the number of observations equal to y_j). In the example above, $p_8(1) = \frac{1}{8}$, $p_8(2) = \frac{2}{8}$, $p_8(4) = \frac{2}{8}$, $p_8(6) = \frac{1}{8}$, $p_8(7) = \frac{1}{8}$, $p_8(9) = \frac{1}{8}$.

The **empirical distribution function** is the distribution function of the empirical random variable that we have just defined:

$$F_n(t) = \frac{\text{number of } x_i\text{'s} \leq t}{n}. \tag{2.19}$$

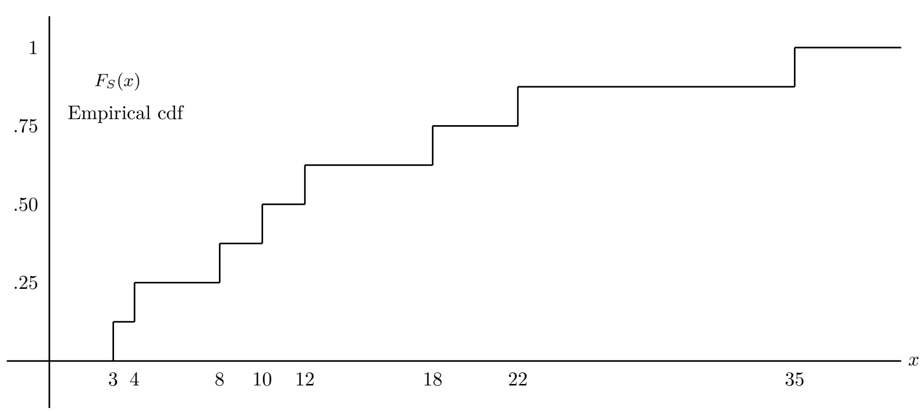
In the example above, $F_8(4) = \frac{5}{8}$. The subscript "8" in F_8 just indicates the total number of data points in the sample.

Example 2.6. A random sample of $n = 8$ values from distribution of X is given: $3, 4, 8, 10, 12, 18, 22, 35$
Formulate the empirical distribution function $F_8(x)$ and draw the graph of $F_{10}(x)$.

Solution.

There are no repeated points. $F_8(t) = \frac{\text{number of } x_i\text{'s} \leq t}{8}$.
The empirical distribution function has values
$F_8(3) = .125\,, F_8(4) = .25\,, F_8(8) = .375\,, \ldots, F_8(22) = .875\,, F_8(35) = 1.0\,.$

The graph of the empirical distribution function $F_8(x)$ is a step function, rising by .125 at each of the sample x-values. The following is the graph of the empirical distribution function.



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2.8 Gamma Function and Related Functions

Many of the continuous distributions described in the FAM Exam Tables make reference to the **gamma function** and the **incomplete gamma function**. The definitions of these functions are

- **gamma function**: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt \quad \text{for} \quad \alpha > 0$

- **incomplete gamma function**: $\Gamma(\alpha;x) = \dfrac{1}{\Gamma(\alpha)} \times \int_0^x t^{\alpha-1}e^{-t}\,dt \quad \text{for} \quad \alpha > 0,\, x > 0$ (2.20)

Some important points to note about these functions are the following:

- if n is an integer and $n \geq 1$, then $\Gamma(n) = (n-1)!$ (2.21)
- $\Gamma(\alpha+1) = \alpha \times \Gamma(\alpha)$ and $\Gamma(\alpha+k) = (\alpha+k-1) \times (\alpha+k-2) \cdots \times \alpha \times \Gamma(\alpha)$ (2.22)

  for any $\alpha > 0$ and integer $k \geq 1$.
- $\int_0^\infty x^k e^{-cx}\,dx = \dfrac{\Gamma(k+1)}{c^{k+1}} \quad \text{for} \quad k \geq 0 \quad \text{and} \quad c > 0$ (use substitution $u = cx$) (2.23)
- $\int_0^\infty \dfrac{1}{x^k}\,e^{-c/x}\,dx = \dfrac{\Gamma(k-1)}{c^{k-1}} \quad \text{for} \quad k > 1 \quad \text{and} \quad c > 0$ (use substitution $u = \frac{c}{x}$) (2.24)

Some of the table distributions make reference to the **incomplete beta function**:

$$\beta(a,b;x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1} \times (1-t)^{b-1}\,dt \quad \text{for} \quad 0 \leq x \leq 1, \quad a,b > 0 \tag{2.25}$$

References to the gamma function have been rare and the incomplete functions have not been referred to on the released exams.

It is useful to remember the integral relationship $\int_0^\infty x^k e^{-cx}\,dx = \dfrac{\Gamma(k+1)}{c^{k+1}}$, particularly in the case in which k is a non-negative integer. In that case, we get $\boldsymbol{\int_0^\infty x^k e^{-cx}\,dx = \dfrac{k!}{c^{k+1}}}$, which can occasionally simplify integral relationships. This relationship is embedded in the definition of the gamma distribution in the FAM Exam Table.

The pdf of the **gamma distribution** with parameters α and θ is $f(t) = \dfrac{t^{\alpha-1}e^{-t/\theta}}{\theta^\alpha\,\Gamma(\alpha)}$, defined on the interval $t > 0$. This means that $\int_0^\infty \dfrac{t^{\alpha-1}e^{-t/\theta}}{\theta^\alpha\,\Gamma(\alpha)}\,dt = 1$, which can be reformulated as $\int_0^\infty t^{\alpha-1}e^{-t/\theta}\,dx = \theta^\alpha \times \Gamma(\alpha)$.

If we let $\theta = \dfrac{1}{c}$ and $k = \alpha - 1$, we get the relationship seen above,

$$\int_0^\infty x^k e^{-cx}\,dx = \frac{\Gamma(k+1)}{c^{k+1}} \tag{2.26}$$

Looking at the various continuous distributions in the FAM Exam Table gives some hints at calculating a number of integral forms. For instance, the pdf of the beta distribution with parameters $a, b, \theta = 1$ is

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times x^{a-1}(1-x)^{b-1} \quad \text{for} \quad 0 < x < 1.$$

Therefore, $\int_0^1 \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times x^{a-1}(1-x)^b\,dx = 1$, from which we get

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Section 2 Problem Set

Preliminary Review - Random Variables I

1. Let X be a discrete random variable with probability function

   $P[X=x]=\dfrac{2}{3^x}$ for $x=1,2,3,\ldots$. What is the probability that X is even?

   (A) $\frac{1}{4}$ (B) $\frac{2}{7}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

2. For a certain discrete random variable on the non-negative integers, the probability function satisfies the relationships $P(0)=P(1)$ and $P(k+1)=\dfrac{1}{k}\times P(k)$ for $k=1,2,3,\ldots$

   Find $P(0)$.

   (A) $\ln e$ (B) $e-1$ (C) $(e+1)^{-1}$ (D) e^{-1} (E) $(e-1)^{-1}$

3. Let X be a continuous random variable with density function

   $f(x)=\begin{cases} 6x(1-x) & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$. Calculate $P[|X-\frac{1}{2}|>\frac{1}{4}]$.

   (A) 0.0521 (B) 0.1563 (C) 0.3125 (D) 0.5000 (E) 0.8000

4. Let X be a random variable with distribution function

   $F(x)=\begin{cases} 0 & \text{for } x<0 \\ \frac{x}{8} & \text{for } 0\le x<1 \\ \frac{1}{4}+\frac{x}{8} & \text{for } 1\le x<2 \\ \frac{3}{4}+\frac{x}{12} & \text{for } 2\le x<3 \\ 1 & \text{for } x\ge 3 \end{cases}$. Calculate $P[1\le X\le 2]$.

   (A) $\frac{1}{8}$ (B) $\frac{3}{8}$ (C) $\frac{7}{16}$ (D) $\frac{13}{24}$ (E) $\frac{19}{24}$

5. Let X_1,X_2 and X_3 be three independent continuous random variables each with density function

   $f(x)=\begin{cases} \sqrt{2}-x & \text{for } 0<x<\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$.

   What is the probability that exactly 2 of the 3 random variables exceeds 1?

   (A) $\frac{3}{2}-\sqrt{2}$ (B) $3-2\sqrt{2}$ (C) $3(\sqrt{2}-1)(2-\sqrt{2})^2$

   (D) $\left(\frac{3}{2}-\sqrt{2}\right)^2\left(\sqrt{2}-\frac{1}{2}\right)$ (E) $3\left(\frac{3}{2}-\sqrt{2}\right)^2\left(\sqrt{2}-\frac{1}{2}\right)$

6. Let X_1, X_2 and X_3 be three independent, identically distributed random variables each with density function $f(x) = \begin{cases} 3x^2 & \text{for } \; 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

Let $Y = \max\{X_1, X_2, X_3\}$. Find $P\left[Y > \frac{1}{2}\right]$.

(A) $\frac{1}{64}$ (B) $\frac{37}{64}$ (C) $\frac{343}{512}$ (D) $\frac{7}{8}$ (E) $\frac{511}{512}$

7. Let the distribution function of X for $x > 0$ be $F(x) = 1 - \sum_{k=0}^{3} \frac{x^k e^{-x}}{k!}$.

What is the density function of X for $x > 0$?

(A) e^{-x} (B) $\frac{x^2 e^{-x}}{2}$ (C) $\frac{x^3 e^{-x}}{6}$ (D) $\frac{x^3 e^{-x}}{6} - e^{-x}$ (E) $\frac{x^3 e^{-x}}{6} + e^{-x}$

8. Let X have the density function $f(x) = \frac{3x^2}{\theta^3}$ for $0 < x < \theta$, and $f(x) = 0$, otherwise.

If $P[X > 1] = \frac{7}{8}$, find the value of θ.

(A) $\frac{1}{2}$ (B) $\left(\frac{7}{8}\right)^{1/3}$ (C) $\left(\frac{8}{7}\right)^{1/3}$ (D) $2^{1/3}$ (E) 2

9. A large wooden floor is laid with strips 2 inches wide and with negligible space between strips. A uniform circular disk of diameter 2.25 inches is dropped at random on the floor. What is the probability that the disk touches three of the wooden strips?

(A) $\frac{1}{\sqrt{\pi}}$ (B) $\frac{1}{\pi}$ (C) $\frac{1}{4}$ (D) $\frac{1}{8}$ (E) $\frac{1}{\pi^2}$

10. If X has a continuous uniform distribution on the interval from 0 to 10, then what is $P\left[X + \frac{10}{X} > 7\right]$?

(A) $\frac{3}{10}$ (B) $\frac{31}{70}$ (C) $\frac{1}{2}$ (D) $\frac{39}{70}$ (E) $\frac{7}{10}$

11. For a loss distribution where $x \geq 2$, you are given:

i) The hazard rate function: $h(x) = \frac{z^2}{2x}$, for $x \geq 2$

ii) A value of the distribution function: $F(5) = 0.84$

Calculate z.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Section 2 Problem Set Solutions

1. $P[X \text{ is even}] = P[X=2]+P[X=4]+P[X=6]+\cdots = \frac{2}{3}\times[\frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^5}+\cdots] = \frac{2}{3^2}\times\frac{1}{1-\frac{1}{3^2}} = \frac{1}{4}$

Answer A

2. $P(2)=P(1)=P(0), \quad P(3)=\frac{1}{2}\times P(2)=\frac{1}{2!}\times P(0),\ldots P(k)=\frac{1}{(k-1)!}\times P(0).$

The probability function must satisfy the requirement $\sum_{i=0}^{\infty} P(i)=1$ so that

$$P(0)+\sum_{i=1}^{\infty}\frac{1}{(i-1)!}\times P(0)=P(0)(1+e)=1$$

(this uses the series expansion for e^x at $x=1$). Then, $P(0)=\dfrac{1}{e+1}$.

Answer C

3. $P\Big[\Big|X-\dfrac{1}{2}\Big|\le\dfrac{1}{4}\Big]=P[-\dfrac{1}{4}\le X-\dfrac{1}{2}\le\dfrac{1}{4}]=P\Big[\dfrac{1}{4}\le X\le\dfrac{3}{4}\Big]=\int_{1/4}^{3/4}6x(1-x)\,dx=.6875$

$\implies P\Big[\Big|X-\dfrac{1}{2}\Big|>\dfrac{1}{4}\Big]=1-P\Big[\Big|X-\dfrac{1}{2}\Big|\le\dfrac{1}{4}\Big]=0.3125.$

Answer C

4. $P[1\le X\le 2]=P[X\le 2]-P[X<1]=F(2)-\lim_{x\to 1^-}F(x)=\dfrac{11}{12}-\dfrac{1}{8}=\dfrac{19}{24}.$

Answer E

5. $P[X\le 1]=\int_0^1(\sqrt{2}-x)\,dx=\sqrt{2}-\dfrac{1}{2} \quad P[X>1]=1-P[X\le 1]=\dfrac{3}{2}-\sqrt{2}.$

With 3 independent random variables, X_1,X_2 and X_3, there are 3 ways in which exactly 2 of the X_i's exceed 1 (either X_1,X_2 or X_1,X_3 or X_2,X_3).

Each way has probability $(P[X>1])^2\times P[X\le 1]=\Big(\dfrac{3}{2}-\sqrt{2}\Big)^2\Big(\sqrt{2}-\dfrac{1}{2}\Big)$

for a total probability of $3\times\Big(\dfrac{3}{2}-\sqrt{2}\Big)^2\Big(\sqrt{2}-\dfrac{1}{2}\Big)$.

Answer E

6. $P\Big[Y>\dfrac{1}{2}\Big]=1-P\Big[Y\le\dfrac{1}{2}\Big]=1-P\Big[\Big(X_1\le\dfrac{1}{2}\Big)\cap\Big(X_2\le\dfrac{1}{2}\Big)\cap\Big(X_3\le\dfrac{1}{2}\Big)\Big]$

$$=1-\Big(P\Big[X\le\frac{1}{2}\Big]\Big)^3=1-\Big[\int_0^{1/2}3x^2\,dx\Big]^3=1-\Big(\frac{1}{8}\Big)^3=\frac{511}{512}$$

Answer E

7. $f(x)=F'(x)=-\sum_{k=0}^{3}\dfrac{kx^{k-1}e^{-x}-x^ke^{-x}}{k!}=e^{-x}\times\sum_{k=0}^{3}\Big[\dfrac{x^k-kx^{k-1}}{k!}\Big].$

$$=e^{-x}\times\Big[1+\frac{x-1}{1}+\frac{x^2-2x}{2}+\frac{x^3-3x^2}{6}\Big]=\frac{e^{-x}x^3}{6}$$

Answer C

8. Since $f(x) = 0$ if $x > \theta$, and since $P[X > 1] = \dfrac{7}{8}$, we must conclude that $\theta > 1$.

Then, $P[X > 1] = \displaystyle\int_1^\theta f(x)\,dx = \int_1^\theta \frac{3x^2}{\theta^3}\,dx = 1 - \frac{1}{\theta^3} = \frac{7}{8}$, or equivalently, $\theta = 2$.

Answer E

9. Let us focus on the left-most point p on the disk. Consider two adjacent strips on the floor. Let the interval [0,2] represent the distance as we move across the left strip from left to right. If p is between 0 and 1.75, then the disk lies within the two strips.

If p is between 1.75 and 2, the disk will lie on 3 strips (the first two and the next one to the right). Since any point between 0 and 2 is equally likely as the left most point p on the disk (i.e. uniformly distributed between 0 and 2) it follows that the probability that the disk will touch three strips is $\dfrac{0.25}{2} = \dfrac{1}{8}$.

Answer D

10. Since the density function for X is $f(x) = \frac{1}{10}$ for $0 < x < 10$, we can regard X as being positive. Then

$$\begin{aligned} P\left[X + \frac{10}{X} > 7\right] &= P[X^2 - 7X + 10 > 0] = P[(X-5)(X-2) > 0] \\ &= P[X > 5] + P[X < 2] = \frac{5}{10} + \frac{2}{10} = \frac{7}{10} \end{aligned}$$

(since $(t-5)(t-2) > 0$ if either both $t-5, t-2 > 0$ or both $t-5, t-2 < 0$)

Answer E

11. The survival function $S(y)$ for a random variable can be formulated in terms of the hazard rate function: $S(y) = exp[-\int_{-\infty}^{y} h(x)\ dx]$.

In this question, $S(5) = 1 - F(5) = 0.16 = \exp\left[-\displaystyle\int_2^5 \frac{z^2}{2x}\,dx\right] = \exp\left[-\frac{z^2}{2}\ln\left(\frac{5}{2}\right)\right]$.

Taking natural log of both sides of the equation results in $-\dfrac{z^2}{2}\ \ln\left(\dfrac{5}{2}\right) = \ln(0.16)$, and solving for z results in $z = 2$.

Answer A

Practice Exam 1

1. Each of decision-makers X, Y, and Z has the opportunity to participate in a game with payoff uniformly distributed on (0, 10000). Assume that X, Y and Z value assets of amount $w \geq 0$ according to the following utility functions:

Decision Maker	Utility Function
X	$u_X(w) = \sqrt{w}$
Y	$u_Y(w) = \frac{w}{100}$
Z	$u_Z(w) = \left(\frac{w}{100}\right)^2$

Which decision maker would not be willing to pay more than 5,000 to participate in the game?

(A) X and Y only

(B) X and Z only

(C) Y and Z only

(D) X, Y and Z

(E) The correct answer is not given by (A), (B), (C) or (D)

2. The XYZ Insurance Company sells property insurance policies with a deductible of \$5,000, policy limit of \$500,000, and a coinsurance factor of 80%. Let X_i be the individual loss amount of the ith claim and Y_i be the claim payment of the ith claim. Which of the following represents the relationship between X_i and Y_i?

(A) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 625{,}000 \\ 500{,}000 & X_i > 625{,}000 \end{cases}$$

(B) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 4{,}000) & 4{,}000 < X_i \leq 500{,}000 \\ 500{,}000 & X_i > 500{,}000 \end{cases}$$

(C) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 630{,}000 \\ 500{,}000 & X_i > 630{,}000 \end{cases}$$

(D) $$Y_i = \begin{cases} 0 & X_i \leq 6{,}250 \\ 0.80\,(X_i - 6{,}250) & 6{,}250 < X_i \leq 631{,}500 \\ 500{,}000 & X_i > 631{,}500 \end{cases}$$

(E) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 505{,}000 \\ 500{,}000 & X_i > 505{,}000 \end{cases}$$

3. A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts can be any non-negative integer, 1, 2, 3, ... without limit. The probability that any given payout is equal to $i > 0$ is $\frac{1}{2^i}$. Payouts are independent. Calculate the probability that there are no payouts of 1,2, or 3 in a given 20 minute period.

(A) 0.08 (B) 0.13 (C) 0.18 (D) 0.23 (E) 0.28

4. Zoom Buy Tire Store, a nationwide chain of retail tire stores, sells 2,000,000 tires per year of various sizes and models. Zoom Buy offers the following road hazard warranty: “If a tire sold by us is irreparably damaged in the first year after purchase, we’ll replace it free, regardless of the cause.”

The average annual cost of honoring this warranty is $10,000,000, with a standard deviation of $40,000. Individual claim counts follow a binomial distribution, and the average cost to replace a tire is $100. All tires are equally likely to fail in the first year, and tire failures are independent. Calculate the standard deviation of the replacement cost per tire.

(A) Less than $60

(B) At least $60, but less than $65

(C) At least $65, but less than $70

(D) At least $70, but less than $75

(E) At least $75

5. A compound Poisson claim distribution has $\lambda = 3$ and individual claims amounts distributed as follows:

x	$f_X(x)$
5	0.6
10	0.4

Determine the expected cost of an aggregate stop-loss insurance with a deductible of 6.

(A) Less than 15.0

(B) At least 15.0 but less than 15.3

(C) At least 15.3 but less than 15.6

(D) At least 15.6 but less than 15.9

(E) At least 15.9

Use the following information for questions 6 and 7. You are the producer of a television quiz show that gives cash prizes. The number of prizes, N, and prize amounts, X are independent of one another and have the following distributions:

N: $P[N=1]=0.8$, $P[N=2]=0.2$

X: $P[X=0]=0.2$, $P[X=100]=0.7$, $P[X=1000]=0.1$

6. Your budget for prizes equals the expected prizes plus 1.25× standard deviation of prizes. Calculate your budget.

(A) 384 (B) 394 (C) 494 (D) 588 (E) 596

7. You buy stop-loss insurance for prizes with a deductible of 200. The cost of insurance includes a 175% relative security load (the relative security load is the percentage of expected payment that is added). Calculate the cost of the insurance.

(A) 204 (B) 227 (C) 245 (D) 273 (E) 357

8. An actuary determines that claim counts follow a negative binomial distribution with unknown β and r. It is also determined that individual claim amounts are independent and identically distributed with mean 700 and variance 1,300. Aggregate losses have a mean of 48,000 and variance 80 million. Calculate the values for β and r.

(A) $\beta=1.20, r=57.19$
(B) $\beta=1.38, r=49.75$
(C) $\beta=2.38, r=28.83$
(D) $\beta=1{,}663.81, r=0.04$
(E) $\beta=1{,}664.81, r=0.04$

9. Let X_1, X_2, X_3 be independent Poisson random variables with means $\theta, 2\theta$, and 3θ respectively. What is the maximum likelihood estimator of θ based on sample values x_1, x_2, and x_3 from the distributions of X_1, X_2 and X_3, respectively,

(A) $\dfrac{\bar{x}}{2}$
(B) $\bar{x}$
(C) $\dfrac{x_1+2x_2+3x_3}{6}$
(D) $\dfrac{3x_1+2x_2+x_3}{6}$
(E) $\dfrac{6x_1+3x_2+2x_3}{11}$

10. Loss random variable X has a uniform distribution on $(0,\theta)$. A sample is taken of n insurance payments from policies with a limit of 100. Eight of the sample values are limit payments of 100. The maximum likelihood estimate of θ is $\widehat{\theta}$. Another sample is taken, also of n insurance payments, but from policies with a limit of 150. Three of the sample values are limit payments of 150. The maximum likelihood estimate of θ is $\frac{4}{3}\widehat{\theta}$.

Determine n.

(A) 40 (B) 42 (C) 44 (D) 46 (E) 48

11. For a group of policies, you are given:

(i) Losses follow a uniform distribution on the interval $(0, \theta)$, where $\theta > 25$.

(ii) A sample of 20 losses resulted in the following:

Interval	Number of Losses
$x \leq 10$	n_1
$10 < x \leq 25$	n_2
$x > 25$	n_3

The maximum likelihood estimate of θ can be written in the form $25 + y$. Determine y.

(A) $\dfrac{25n_1}{n_2 + n_3}$

(B) $\dfrac{25n_2}{n_1 + n_3}$

(C) $\dfrac{25n_3}{n_1 + n_2}$

(D) $\dfrac{25n_1}{n_1 + n_2 + n_3}$

(E) $\dfrac{25n_2}{n_1 + n_2 + n_3}$

12. The number of claims follows a negative binomial distribution with parameters β and r, where β is unknown and r is known. You wish to estimate β based on n observations, where $\bar{x}$ is the mean of these observations. Determine the maximum likelihood estimate of β.

(A) $\dfrac{\bar{x}}{r^2}$ (B) $\dfrac{\bar{x}}{r}$ (C) $\bar{x}$ (D) $r\bar{x}$ (E) $r^2\bar{x}$

13. The following 6 observations are assumed to come from the continuous distribution with pdf $f(x;\theta) = \frac{1}{2}x^2\theta^3 e^{-\theta x}$: 1, 3, 4, 4, 5, 7.

Find the mle of θ.

(A) 0.25 (B) 0.50 (C) 0.75 (D) 1.00 (E) 1.25

14. An analysis of credibility premiums is being done for a particular compound Poisson claims distribution, where the criterion is that the total cost of claims is within 5% of the expected cost of claims with a probability of 90%. It is found that with $n = 60$ exposures (periods) and $\bar{X} = 180.0$, the credibility premium is 189.47. After 20 more exposures (for a total of 80) and revised $\bar{X} = 185$, the credibility premium is 190.88. After 20 more exposures (for a total of 100) the revised $\bar{X}$ is 187.5. Assuming that the manual premium remains unchanged in all cases, and assuming that full credibility has not been reached in any of the cases, find the credibility premium for the 100 exposure case.

(A) 191.5 (B) 192.5 (C) 193.5 (D) 194.5 (E) 196.5

15. For an insurance portfolio, you are given:

(i) For each individual insured, the number of claims follows a Poisson distribution.

(ii) The mean claim count varies by insured, and the distribution of mean claim counts follows gamma distribution.

(iii) For a random sample of 1000 insureds, the observed claim counts are as follows:

Number of Claims, n	0	1	2	3	4	5
Number of Insureds, f_n	512	307	123	41	11	6

$$\sum n f_n = 750, \qquad \sum n^2 f_n = 1494$$

(iv) Claim sizes follow a Pareto distribution with mean 1500 and variance 6,750,000.

(v) Claim sizes and claim counts are independent.

(vi) The full credibility standard is to be within 5% of the expected aggregate loss 95% of the time.

Determine the minimum number of insureds needed for the aggregate loss to be fully credible.

(A) Less than 8300
(B) At least 8300, but less than 8400
(C) At least 8400, but less than 8500
(D) At least 8500, but less than 8600
(E) At least 8600

Information on Questions 16 and 17 is as follows. You are given the following information on cumulative incurred losses through development years shown.

	Cumulative Incurred Losses				Paid-to-Date
Accident	Development Year				at Dec 31, AY4
Year	0	1	2	3	
AY1	2325	3749	4577	4701	4701
AY2	2657	4438	5529		4500
AY3	2913	4995			3500
AY4	3163				2500

16. Using an average factor model, calculate the estimated total loss reserve as of Dec. 31, AY4.

(A) Less than 7500
(B) At least 7500 but less than 7800
(C) At least 7800 but less than 8100
(D) At least 8100 but less than 8400
(E) At least 8400

17. As of Dec. 31, AY4, calculate

Estimated reserve for AY2 based on an average factor model)
− (Estimated reserve for AY2 based on a mean factor model)

(A) Less than −300

(B) At least −300 but less than −100

(C) At least −100 but less than 100

(D) At least 100 but less than 300

(E) At least 300

18. For a one-period binomial model for the price of a stock with price 100 at time 0, you are given:

(i) The stock pays no dividends.

(ii) The stock price is either 110 or 95 at the end of the year.

(iii) The risk free force of interest is 5%.

Calculate the price at time 0 of a one-year call option with strike price 100.

(A) Less than 6.00

(B) At least 6.00 but less than 6.25

(C) At least 6.25 but less than 6.50

(D) At least 6.50 but less than 6.75

(E) At least 6.75

19. Using the following information, determine the incurred losses for the 2017 accident year as reported at Dec. 31, 2018.

Occurrence #1: Occurrence date Feb. 1/16, Report date Apr. 1/16

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Apr. 1/16	1000	1000	2000
Dec. 31/16	1500	1000	2500
Dec. 31/17	1500	1000	2500
Mar. 1/18	3000	0	3000

Occurrence #2: Occurrence date May 1/17, Report date July 1/17

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
July 1/17	1000	2000	3000
Dec. 31/17	3000	1000	4000
Dec. 31/18	5000	0	5000

Occurrence #3: Occurrence date Nov. 1/17, Report date Feb. 1/18

Loss History

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Mar. 1/18	0	8000	8000
Dec. 31/18	5000	5000	10,000

(A) 0
(B) 3,000
(C) 5,000
(D) 8,000
(E) 15,000

20. You are given the following calendar year earned premium.

Year	CY2	CY3	CY4
Earned Premium	4200	4700	5000

You are also given the following rate changes

Date	April 1, CY1	September 1, CY2	July 1, CY3
Average Rate Change	+12 %	+6 %	+10 %

Determine the approximate earned premium at current (end of CY4) rates for CY3.

(A) Less than 5000
(B) At least 5000 but less than 5100
(C) At least 5100 but less than 5200
(D) At least 5200 but less than 5300
(E) At least 5300

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1. A decision maker with utility function $u(w)$ will pay amount C to play a game (random variable) W based on the following relationship: $u(C) = E[u(W)]$.

For individual X: $E[u_X(W)] = \int_0^{10{,}000} \frac{\sqrt{w}}{10{,}000}\, dw = \frac{200}{3} = u_X(C_X) = \sqrt{C_X}$.

Solving for C_X results in $C_X = \left(\frac{200}{3}\right)^2 = 4{,}444.44$

For individual Y: $E[u_Y(W)] = \int_0^{10{,}000} \frac{w}{10{,}000} \times \frac{1}{10{,}000}\, dw = 50 = u_Y(C_Y) = \frac{C_Y}{100}$.

Solving for C_Y results in $C_Y = 5{,}000$

For individual Z: $E[u_Z(W)] = \int_0^{10{,}000} \left(\frac{w}{10{,}000}\right)^2 \times \frac{1}{10{,}000}\, dw = \frac{100}{3} = u_Z(C_Z) = \left(\frac{C_Z}{100}\right)^2$.

Solving for C_Z results in $C_Z = 1000 \times \sqrt{\frac{100}{3}} = 5{,}773.50$

Only Z will pay more than 5,000 for the gamble. In general the concavity of the utility function determines the risk profile of the individual. X has a concave utility function $(u_X''(w) < 0)$ and will be risk-averse and will not be willing to pay more that the expected value (fair cost) of the gamble of 5,000. Y has a linear utility function and will be risk-neutral. Y will be willing to pay at most the expected payoff of the gamble of 5.000 but not more. Z has a convex utility function $(u_X''(w) > 0)$ and will be risk-preferring and will pay more the 5,000 for the gamble.

Answer A

2. With coinsurance factor α, deductible d, policy limit $\alpha(u-d)$, the amount paid per loss is

(we are assuming in inflation rate of $r = 0$)$Y = \begin{cases} 0 & X \le d \\ \alpha(X-d) & d < X \le u. \\ \alpha(u-d) & X > u \end{cases}$

In this problem, the coinsurance factor is $\alpha = .8$, the deductible is $d = 5{,}000$, and the policy limit is $.8(u - 5{,}000) = 500{,}000$, so that the maximum covered loss is $u = 630{,}000$.

The amount paid per loss becomes $Y = \begin{cases} 0 & X \le 5{,}000 \\ 0.80(X - 5{,}000) & 5{,}000 < X \le 630{,}000 \\ 500{,}000 & X > 630{,}000 \end{cases}$

Answer C

3. When a payout occurs, it is 1, 2 or 3 with probability $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{8}$. The number of payouts that are 1,2 or 3 follows a Poisson process with an hourly rate of $5 \times \frac{7}{8} = \frac{35}{8}$.
The expected number of payouts that are 1, 2 or 3 in 20 minutes, say N, has a Poisson distribution with mean $\frac{35}{8} \times \frac{20}{60} = \frac{35}{24}$. The probability that there are no payouts of 1, 2, or 3 in a given 20 minute period is the probability that $N = 0$, which is $e^{-35/24} = .233$.

Answer D